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MATHEMATICS

Pure Mathematics
Unit P2

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Pure Mathematics Unit P2

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PREFACE

This text is the second of three volumes which will cover between them most of the mathematical methods required for a modular A level course in mathematics. Specifically, the text is based on the P2 Specification of the Welsh Joint Education Committee which was introduced in September 2000.

It is assumed that the reader will have successfully completed a GCSE course in mathematics and will have access to a calculator possessing mathematical functions.

The text concludes with six revision papers. It is believed that these tests should be completed in approximately one hour by students who are ready to sit their A level examinations.

CONTENTS

Chapter 1

Solution of Inequalities

The solution of equations was considered in **P1**. There, the concept of inequalities arose during consideration of the discriminant of a quadratic function. Here, we take a more detailed look at inequalities.

1.1 Inequalities

Often in mathematics we are asked to consider relationships such as

$$2b + 6 > 4$$

or $$c^2 - 2c < 6$$

or $$2y^2 + 3y + 5 \geq 0$$ and so on.

The symbol $>$ means greater than,

$<$ means less than,

and $\geq$ means greater than or equal to.

Note that the sharp end of the arrows $>$ and $<$ always point to the smaller number.

Thus $6 > 5$ and $3 < 4$.

Exercises 1.1

Use $>$, $<$ and $\geq$ to write the following statements :-

a) 12 is greater than 9.
b) 4 is less than 7.
c) x is greater than or equal to y.
d) m is positive.
e) p is not negative.

Relationships involving $>$ and $<$ are called **inequalities**. When the possibility of equality is not allowed (i.e. $>$ and $<$ rather than $\geq$ or $\leq$) the inequality is said to be a **strict inequality**.

When inequalities involve letters, e.g. $2z + 6 > 10$, our usual interest is to find the range of values of the letter (z in above) in order for the inequality to be satisfied; in other words, we are interested in solving the inequality. In this book we shall consider the solution of linear or quadratic inequalities, for example

$$6 - 4a < 3,$$

and $$2x^2 + 9x + 7 \geq 0.$$

We start with linear inequalities. The rules given in **P1** relating to equations can be modified for use with inequalities.

Manipulative rules for use with inequalities
In the following, a, b, c and d are real numbers.

(i) If $a > b$
then $a - b > 0$.

> $6 > 2$
> $6 - 2 > 2 - 2$
> i.e. $4 > 0$.

(ii) If $a > b$
then $a + c > b + c$
and $a - d > b - d$.

> $7 > 3$
> $7 + 2 > 3 + 2$
> i.e., $9 > 5$

> $8 > 5$
> $8 - 3 > 5 - 3$
> i.e., $5 > 2$

(iii) If $ad > bd$ and d is positive ($d > 0$)
then $a > b$.

> $12 > 9$
> so $\frac{12}{3} > \frac{9}{3}$
> i.e., $4 > 3$

(iv) If $ad > bd$ and d is negative ($d < 0$)
then $a < b$.

> $6 > -4$
> $\frac{6}{-2} < \frac{-4}{-2}$
> i.e. $-3 < 2$.

(v) If $\frac{a}{d} > b$ and d is positive ($d > 0$)
then $a > bd$.

> $\frac{36}{9} > 3$
> so $36 > 27$

(vi) If $\frac{a}{d} > b$ and d is negative ($d < 0$)
then $a < bd$.

> $\frac{-24}{-3} > 5$
> so $-24 < -15$

(vii) If $ab > 0$
then $a > 0$ <u>and</u> $b > 0$
<u>or</u> $a < 0$ <u>and</u> $b < 0$.

> **The product of two negative numbers or two positive numbers is positive.**

(viii) If $ab < 0$
then $a < 0$ <u>and</u> $b > 0$
<u>or</u> $a > 0$ <u>and</u> $b < 0$.

> **The product of a positive number and a negative number is negative.**

Rules (iii) – (vi) are particularly important and often lead to errors in problems. Essentially, multiplication (or division) throughout an inequality results in no change in the direction of the inequality (> leads to >, < leads to <) if the number multiplying or dividing is positive.
However, if the multiplication or division involves a negative number, the direction of the inequality is reversed (i.e. > leads to < and < leads to >).

Example 1.1
Solve the inequality (find the range of values of x)

$$3x - 6 < 4.$$

The general strategy is similar to that for linear equations: isolate x on one side.

$\therefore \quad 3x < 4 + 6$ (Rule (ii), add 6 to both sides)

so $\quad 3x < 10.$

$\therefore \quad x < \frac{10}{3}.$ (Rule (iii), division by positive number)

Example 1.2
Solve $\quad 2x - 4 \geq 7x - 2.$

Then $\quad 2x - 4 - 7x \geq -2$ (Rule (ii), subtract $7x$ from both sides)

so $\quad -5x - 4 \geq -2.$

$\therefore \quad -5x \geq -2 + 4$ (Rule (ii), add 4 to both sides)

and $\quad -5x \geq 2.$

$\therefore \quad x \leq -\frac{2}{5}.$ $\left(\begin{array}{l}\text{Rule (iv), division by negative} \\ \text{number } (-5) \text{ reverses inequality}\end{array}\right)$

The rules need not be stated as in examples 1.1, 1.2. However, until the reader is familiar with them, he/she is advised to justify each step as shown.
The method of solution of quadratic inequalities also makes use of Rule (viii).

Example 1.3
Solve $\quad x^2 - 6x + 8 > 0.$

The left hand factorises so that

$$(x-4)(x-2) > 0.$$

Then either $\quad x - 4 > 0$ and $x - 2 > 0$ (Rule vii)

or $\quad x - 4 < 0$ and $x - 2 < 0.$

Thus either $\quad x > 4$ and $x > 2$ i.e. $x > 4$

or $\quad x < 4$ and $x < 2$ i.e. $x < 2.$

Combination of both statements thus gives the solution as $x > 4$ or $x < 2$.

Example 1.4
Solve $\quad x^2 - x - 4 \leq x + 4.$

We move all terms to one side

$$\therefore \quad x^2 - x - 4 - x - 4 \leq 0$$

giving $x^2 - 2x - 8 \leq 0.$

Rule (i) with > replaced by ≤ .

The left hand factorises.

$$\therefore (x-4)(x+2) \leq 0.$$

From Rule (viii), one factor is positive, one negative. The possibility of equality is allowed for by the word 'inclusive' in the last line below.

Case 1 $\quad x - 4 > 0$ and $x + 2 < 0$

$\therefore \quad x > 4$ and $x < -2$.
This is impossible.

Ignore possibility of equality, see last line.

Case 2 $\quad x - 4 < 0$ and $x + 2 > 0$

$\therefore \quad x < 4$ and $x > -2$

$\therefore \quad -2 \le x \le 4$,

i.e. x lies between -2 and 4 (inclusive).

Remember now to allow for possible equality.

Sometimes the quadratic will not factorise, in which case we resort to using completion of the square.

Example 1.5

Solve $\quad x^2 + 8x - 18 < 0$.

The left hand side does not factorise, and we complete the square.

See P1 for completion of square.

Then $\quad (x + 4)^2 - 16 - 18 < 0$.

$\therefore \quad (x + 4)^2 - 34 < 0$,

so $\quad (x + 4)^2 < 34$.

Now $\quad -\sqrt{34} < x + 4 < \sqrt{34}$.

$\therefore \quad -\sqrt{34} - 4 < x < \sqrt{34} - 4$.

Any number between $-\sqrt{34}$ and $\sqrt{34}$ has a square less than +34.

The values of x must lie between
$-\sqrt{34} - 4$ and $\sqrt{34} - 4$.

Example 1.6

Solve $\quad x^2 - 4x - 10 \ge 0$.

Completion of the square gives

$$(x - 2)^2 - 4 - 10 \ge 0$$

so $\quad (x - 2)^2 \ge 14$.

Then $(x - 2) \ge \sqrt{14}$ or $(x - 2) \le -\sqrt{14}$.

The first gives $\quad x \ge 2 + \sqrt{14}$,

the second gives $\quad x \le 2 - \sqrt{14}$.

$\therefore$ The solution is $\quad x \ge 2 + \sqrt{14}$

or $\quad x \le 2 - \sqrt{14}$.

Inequalities often occur as part of a bigger problem.

Example 1.7

Find the range of values of d such that the quadratic equation (in x)

$$(d + 2)x^2 - 2dx + 1 = 0 \quad \text{has real roots.}$$

Comparing with the standard quadratic, we see that

$a = d + 2, b = -2d, c = 1.$

A quadratic $ax^2 + bx + c = 0$ has real roots if $b^2 \geq 4ac$.

Then $(-2d)^2 - 4(1)(d+2) \geq 0$

so $4d^2 - 4(d+2) \geq 0$

or $d^2 - (d+2) \geq 0.$ (Rule (iii) with division by 4)

$\therefore$ $d^2 - d - 2 \geq 0.$

Factorise, $(d+1)(d-2) \geq 0.$

Then $d + 1 \geq 0$ <u>and</u> $d - 2 \geq 0$

or $d + 1 \leq 0$ <u>and</u> $d - 2 \leq 0.$ (Rule vii)

The first pair of inequalities gives $d \geq 2$

and the second pair of inequalities gives $d \leq -1$.

Thus the quadratic equation has real roots if $d \leq -1$ or $d \geq 2$.

Exercises 1.2

1. Find the range of values of x satisfying the following linear inequalities :-

(i) $6 < 2 + x$ (ii) $7 > 3 - x$ (iii) $4 - x < 6x$

(iv) $5 - 2x > 3x + 2$ (v) $7 - 2x > -5 - 3x$

(vi) $3(x-1) \leq 2$ (vii) $3(x+5) > 2(x-3)$

(viii) $2(x-3) - 3(x-1) \leq 4(x+1) - 7(x-3)$

2. Find the range of values of x satisfying the following :-

(i) $x^2 + 6x + 5 > 0$ (ii) $x^2 - 9x + 20 < 0$

(iii) $x^2 + 6x - 7 \geq 0$ (iv) $x^2 + 18x + 72 \leq 0$

(v) $2x^2 - 5x + 1 > 0$ (vi) $5x^2 - 18x + 16 < 0$

(vii) $5x^2 - 2x - 16 > 0$ (viii) $2x^2 - 5x - 1 \leq 0$

(ix) $3x^2 - 7x + 3 \geq 0$ (x) $4x^2 + 5x - 1 \leq 0$

(xi) $x^2 + x + 3 < 2x + 5$

3. Find the range of values of k such that

(i) $(k+1)x^2 - 2x - 3 = 0$ has real roots

(ii) $3x^2 - kx + 4 = 0$ has no real roots

(iii) $(2k+1)x^2 + (k+2)x + 1 = 0$ has real roots

(iv) $(k+1)x^2 + 3x - 2 = 0$ has no real roots

(v) $(k+3)x^2 + (2k+1)x + (k+1) = 0$ has real roots.

4. Show that the equation $(2a-1)x^2 - 2ax + 1 = 0$
has real roots whatever the value of a, as long as a is real.

1.2 The Modulus Sign

The modulus of a real number is a measure of its size irrespective of the sign of the number. We denote the modulus by the symbol $|\ |$. Then

$|3| = 3$ but also $|-3| = 3$.

Exercises 1.3

1. Write down the moduli of the following real numbers:-
(i) -2 (ii) 1 (iii) 0 (iv) $4 - 16 - 12$
The effect of the modulus sign is therefore to assign a positive sign to all numbers. The modulus sign sometimes occurs in inequalities.

Example 1.8

Solve the inequality

$$|x - 6| < 4.$$

> If a number has size less than 4 irrespective of sign it must be between –4 and 4.

$\therefore$ $-4 < x - 6 < 4$

so $-4 + 6 < x < 4 + 6.$

$\therefore$ $2 < x < 10.$

> Rule (ii) section 1.1

Example 1.9

Solve $|2x - 7| > 5.$

Then $2x - 7 > 5$

or $2x - 7 < -5.$

> Note $|-8| > 5$, for example.

The first gives $2x > 7 + 5$

so $2x > 12$

and $x > 6.$

The second gives $2x < 7 - 5$

so $2x < 2$

and $x < 1.$

The solution is $x > 6$ or $x < 1.$

Exercises 1.4

1. Solve the inequalities :-
(i) $|x + 7| < 9$ (ii) $|2x - 3| > 6$
(iii) $|5 - 4x| < 6$ (iv) $|3 - 2x| \geq 2$

2. Solve the equations
(i) $|x - 2| = 5$ (ii) $|2x - 3| = 7$
(iii) $|5 - 2x| = 13$ (iv) $|x^2 - 12x + 5| = 3$
(v) $|2y^2 + 4y - 1| = 2$

> Hint: if $|a| = 3$ then $a = 3$ or -3

1.3 Interval Notation

In the previous two sections we expressed solutions to inequalities in the form $x > -5$, $x \leq 7$, $9 \leq x < 12$, $x > 2$ or $x \leq -4$, and so on. It is convenient to write such solutions in interval form.

The various cases may be written as follows :-

(i) $x > -5$ as $(-5, \infty)$,
(ii) $x \geq -5$ as $[-5, \infty)$,
(iii) $x < 7$ as $(-\infty, 7)$,

> '∞' signifies numbers increasing indefinitely. '$-\infty$' signifies numbers decreasing indefinitely.

(iv) $x \le 7$ as $(-\infty, 7]$,
(v) $9 \le x < 12$ as $[9, 12)$,
(vi) $x < 2$ or $x \ge 7$ as $(-\infty, 2) \cup [7, \infty)$,
(vii) $-8 < x < 4$ as $(-8, 4)$.

The round brackets occur in strict inequalities, the square brackets in inequalities which also allow the possibility of equality.
The $\cup$ in $(-\infty, 2) \cup [7, \infty)$ signifies the union of two intervals, in other words all values in either interval.

Exercises 1.5

1. Represent the following in interval form:-
(a) $x > -3$ (b) $x \le 6$ (c) $x \ge 9$ (d) $x < -4$
(e) $-3 \le x < 21$ (f) $x \ge 9$ and $x < 12$
(g) $x > -5$ and $x \le 20$ (h) $x \ge -20$ or $x \le -30$

2. Express the solutions to the inequalities in Exercises 1.2, questions 2(i) – (iv) in interval form.

Chapter 2

Factorising and Two Theorems

In **P1** we considered the factorisation of quadratic expressions, i.e. of polynomials of degree two. Here we consider factorisation of polynomials of higher degree. Factorisation of polynomials may involve division of one polynomial by another. For that reason, we start by considering division of polynomials.

2.1 Division of polynomials

Let's start by considering an example.

Example 2.1.

Divide $x^3 + 7x^2 + 4x - 9$ by $x + 2$.

The method used is similar to the method used when we divide a number by another number without the use of a calculator. We subtract $x + 2$ from $x^3 + 7x^2 + 4x - 9$ as many times as possible.

The terms in $x^3 + 7x^2 + 4x - 9$ are eliminated one by one as follows.

For numbers, division of 38 by 6 is achieved by taking 6 from 38 six times leaving remainder 2.

$$\begin{array}{lr} & x^2 + 5x - 6 \\ x+2\,\big) & x^3 + 7x^2 + 4x - 9 \\ & x^3 + 2x^2 \quad\quad\quad \\ \text{Subtract} & 5x^2 + 4x \quad\; \\ & 5x^2 + 10x \quad\; \\ \text{Subtract} & -6x - 9 \\ & -6x - 12 \\ \text{Subtract} & 3 \end{array}$$

To eliminate x^3 we multiply $x + 2$ by x^2.

Bring down next term and eliminate $5x^2$ by multiplying $x + 2$ by $5x$.

Bring down next term and eliminate $-6x$ by multiplying $x + 2$ by -6.

$-9 - (-12) = -9 + 12 = 3$.

The remainder 3 doesn't contain x so no further elimination is necessary and we are finished.

Then $$\frac{x^3 + 7x^2 + 4x - 9}{x+2} \equiv x^2 + 5x - 6 + \frac{3}{x+2} \qquad (x \neq -2)$$

or equivalently

$$x^3 + 7x^2 + 4x - 9 \equiv (x^2 + 5x - 6)(x + 2) + 3.$$

The following points should be noted.

1. The long division procedure involves attempting to eliminate the highest power of x at each stage.
2. The process ends when no further elimination is possible by multiplication by a power of x or a number.
3. At each step, the signs in the highest power terms are the same and therefore elimination is always achieved by subtraction.

$$\begin{array}{r} -6x-9 \\ \underline{-6x-12} \\ 3 \end{array}$$

Example 2.2

Divide $8x^4 - 24x^3 + 4x^2 - 9x + 5$ by $2x - 3$.

Whilst the first term involves x^4 instead of x^3 as in example 2.1, the same procedure is adopted.

$$\begin{array}{rl}
 & 4x^3 - 6x^2 - 7x - 15 \\
2x-3 \,\big) & 8x^4 - 24x^3 + 4x^2 - 9x + 5 \\
 & \underline{8x^4 - 12x^3} \\
 & -12x^3 + 4x^2 \\
 & \underline{-12x^3 + 18x^2} \\
 & -14x^2 - 9x \\
 & \underline{-14x^2 + 21x} \\
 & -30x + 5 \\
 & \underline{-30x + 45} \\
 & -40
\end{array}$$

eliminate $8x^4$

eliminate $-12x^3$

eliminate $-14x^2$

-40 cannot be eliminated

Then $\dfrac{8x^4 - 24x^3 + 4x^2 - 9x + 5}{2x-3} \equiv 4x^3 - 6x^2 - 7x - 15 - \dfrac{40}{2x-3} (x \neq \dfrac{3}{2})$.

Or $8x^4 - 24x^3 + 4x^2 - 9x + 5 \equiv (4x^3 - 6x^2 - 7x - 15)(2x-3) - 40$.

The procedure may also be used when the bottom expression is of a degree higher than one.

Example 2.3

Divide $2x^4 + 3x^3 + 9x^2 - 5x + 1$ by $x^2 + 6x + 1$.

$$\begin{array}{rl}
 & 2x^2 - 9x + 61 \\
x^2+6x+1 \,\big) & 2x^4 + 3x^3 + 9x^2 - 5x + 1 \\
 & \underline{2x^4 + 12x^3 + 2x^2} \\
 & -9x^3 + 7x^2 - 5x \\
 & \underline{-9x^3 - 54x^2 - 9x} \\
 & 61x^2 + 4x + 1 \\
 & \underline{61x^2 + 366x + 61} \\
 & -362x - 60
\end{array}$$

eliminate $2x^4$

eliminate $-9x^3$

$7x^2 - (-54x^2)$
$= 7x^2 + 54x^2 = 61x^2$

This process ends because $-362x - 60$ is of a degree less than that of $x^2 + 6x + 1$ and no further elimination is possible.

$\therefore \quad 2x^4 + 3x^3 + 9x^2 - 5x + 1 \equiv (x^2 + 6x + 1)(2x^2 - 9x + 61) - 362x - 60.$

Sometimes, one polynomial divides another exactly and no remainder is obtained.

Example 2.4

Show that $2x + 3$ divides exactly into $2x^3 + 5x^2 - 9x - 18$.

$$\begin{array}{r}
x^2 + x - 6 \\
2x + 3 \,\big)\, \overline{2x^3 + 5x^2 - 9x - 18} \\
\underline{2x^3 + 3x^2} \\
2x^2 - 9x \\
\underline{2x^2 + 3x} \\
-12x - 18 \\
\underline{-12x - 18} \\
- - - -
\end{array}$$

Then $\quad 2x^3 + 5x^2 - 9x - 18 \equiv (2x + 3)(x^2 + x - 6).$

Given that a linear factor divides exactly into another polynomial, the division may be carried out by another method.

Example 2.5

Given that $3x^3 - 17x^2 + 37x - 18$ is divided exactly by $3x - 2$, we can write

$$3x^3 - 17x^2 + 37x - 18 = (3x - 2)(x^2 + ax + 9).$$

The expression in the second bracket must be of second degree and must be of the form shown in order that the x^3 and constant terms match.

$3x^3$ _ _ _ _ _ _ $- 18 \equiv (3x - 2)(x^2 + ax + 9)$

$3x^3 \qquad\qquad - 18$

The constant a is unknown but is easily found by consideration of the x^2 or x terms.

<u>x^2 term</u>

$$3x^3 - 17x^2 + 37x - 18 = (3x - 2)(x^2 + ax + 9)$$

The x^2 terms are formed as shown.

Thus	$-17x^2$	$\equiv$	$-2x^2 + 3ax^2$	
so	$-17 + 2$	$=$	$3a$	(cancelling x^2).
$\therefore$	a	$=$	-5,	

or alternatively, for the x term

	$37x$	$\equiv$	$27x - 2ax$	
so	10	$=$	$-2a$	
and	a	$=$	-5,	as before.

$(3x - 2)(x^2 + ax + 9)$

Substitution for a in the suggested factorisation gives

$$3x^3 - 17x^2 + 37x - 18 = (3x - 2)(x^2 - 5x + 9).$$

Example 2.6

Given that $2x - 5$ divides exactly into $2x^4 - 11x^3 + 17x^2 - x - 10$, we may write

$$2x^4 - 11x^3 + 17x^2 - x - 10 \equiv (2x - 5)(x^3 + ax^2 + bx + 2).$$

$$2x^4 \qquad\qquad -10$$

The form of the cubic expression on the right hand side is deduced from consideration of the terms in x^4 ($2x^4$) and the constant term ($-$ 10). The constants a and b are determined from consideration of the terms in x^3, x^2 or x.

<u>The term in x</u>

$-x \equiv 4x - 5bx$

so $-5x \equiv -5bx$

or $b = 1$.

$(2x - 5)(x^3 + ax^2 + bx + 2)$

<u>The term in x^2</u>

$17x^2 \equiv 2x^2 - 5ax^2$

so $15x^2 \equiv -5ax^2$

and $a = -3$.

$(2x - 5)(x^3 + ax^2 + x + 2)$

Then $2x^4 - 11x^3 + 17x^2 - x - 10 \equiv (2x - 5)(x^3 - 3x^2 + x + 2)$.

Consideration of the term in x^3 gives

$-11x^3 \equiv -5x^3 + 2ax^3$

so $-6x^3 \equiv 2ax^3$

and $a = -3$, as before.

There is no need for this but it is given as a check : $(2x - 5)(x^3 + ax^2 + bx + 2)$

Exercises 2.1

1. Derive the relationship between the polynomials A and B in the form
Polynomial A $\equiv$ (polynomial B)(polynomial) + remainder
for the following cases.
   - (i) (A) $x^2 - 3x - 2$ (B) $x + 3$
   - (ii) (A) $x^3 - 3x^2 + 4x - 5$ (B) $x - 5$
   - (iii) (A) $2x^3 - 7x^2 + 6x - 3$ (B) $2x + 1$
   - (iv) (A) $12x^4 - 8x^3 + 21x^2 + 1$ (B) $6x + 5$
   - (v) (A) $12x^4 - x^3 + 12x^2 - 7$ (B) $4x^2 - 3x + 2$
2. Find the expressions denoted by ? in the following :-
   - (i) $3x^3 + 5x^2 - 25x - 7 = (3x - 7)(?)$
   - (ii) $x^3 + 2x^2 - x - 2 = (x + 2)(?)$
   - (iii) $x^4 - 5x^3 + 9x^2 - 7x + 2 = (x^2 - 3x + 2)(?)$
3. Show that $x - 5$ divides $x^3 - 4x^2 - 17x + 60$ exactly and factorise the polynomial.
4. Show that both $x - 2$ and $x - 3$ divide $x^4 - 6x^3 - x^2 + 54x - 72$ exactly and factorise the polynomial.

2.2 Two theorems

Question 3, Exercises 2.1 shows that if one factor of a cubic polynomial is known the other factors are easily found. Specifically,

$$\begin{aligned} f(x) &= x^3 - 4x - 17x + 60 \\ &\equiv (x-5)(x^2+x-12) \\ &\equiv (x-5)(x+4)(x-3). \end{aligned}$$

Clearly, the procedure of factorising polynomials of degree higher than two depends crucially on our knowing a factor to start the process. Much of this section is concerned with finding that important first factor.

Let's recap on our findings of Section 2.1. When a polynomial f(x) is divided by a linear factor $x - a$ we have

$$f(x) \equiv (x-a)\,Q(x) + R,$$

where $Q(x)$ is a polynomial and the Remainder R is a number.

In example 2.1,
$x^3 + 7x^2 + 4x - 9$
$\equiv (x+2)(x^2 + 5x - 6) + 3$

$Q(x)$ is of degree one less than that of f(x) and the number R is, of course, of degree one less than $x - a$.

Then, from above, on setting $x = a$, we have

$$f(a) = (a-a)\,Q(a) + R$$

so that $\quad R = f(a)$.

This result is known as the **Remainder Theorem.**

Remainder Theorem
When a polynomial expression $f(x)$ is divided by the linear expression $x - a$, the resulting remainder is $f(a)$.

Example 2.7

Find the remainder when $x^2 + 3x + 5$ is divided by $x + 2$.

Writing $f(x) = x^2 + 3x + 5$ and noting that $a = -2$, we have

remainder $= f(-2) = (-2)^2 + 3(-2) + 5 = 3$.

Example 2.8

When $x^2 + bx + c$ is divided by $x - 1$, the remainder is –14; when divided by $x + 1$, the remainder is 0. Find b and c.

$$\begin{aligned} \text{If} \qquad f(x) &= x^2 + bx + c, \\ f(1) &= -14, \\ f(-1) &= 0. \\ \therefore \qquad 1 + b + c &= -14, \qquad (1) \\ 1 - b + c &= 0. \qquad (2) \end{aligned}$$

From (1), (2) we find $b = -7$, $c = -8$.

Check

From the remainder theorem, we saw that

$$f(x) = (x - a)Q(x) + R$$
$$\text{with } R = f(a)$$

Now when $x - a$ is a factor of $f(x)$, there will be no remainder when $f(x)$ is divided by $x - a$. Then

$$R = f(a) = 0.$$

This result is known as the **Factor Theorem.**

> <u>Factor Theorem</u>
> If a polynomial expression $f(x)$ is such that $f(a) = 0$, then $x - a$ is a factor or, in other words, $x - a$ divides into $f(x)$ exactly.

The factor theorem is a useful tool for the factorisation of polynomials.

Example 2.9
Factorise $2x^3 - x^2 - 13x - 6$.

If $f(x) = 2x^3 - x^2 - 13x - 6$, we seek a number to substitute for x which makes $f(x)$ equal to zero.

Now $\quad f(0) = 2.(0^3) - (0)^2 - 13(0) - 6 = -6 \neq 0.$

doesn't work

$f(1) = 2 - 1 - 13 - 6 = -18 \neq 0.$
$f(-1) = -2 - 1 + 13 - 6 = 4 \neq 0.$
$f(2) = 2(2)^3 - (2)^2 - 13(2) - 6$
$\quad\;\; = 16 - 4 - 26 - 6 = -20 \neq 0.$

watch the signs: use brackets

$f(-2) = 2(-2)^3 - (-2)^2 - 13(-2) - 6$
$\quad\;\; = -16 - 4 + 26 - 6 = 0.$

Since $f(-2) = 0$, $x + 2$ is a factor.

Then this factor can be divided out to give

$$2x^3 - x^2 - 13x - 6 \equiv (x + 2)(2x^2 - 5x - 3)$$
$$\equiv (x + 2)(x - 3)(2x + 1),$$

on factorising the quadratic expression in the usual way (see **P1**).

$$\begin{array}{r|l} & 2x^2 - 5x - 3 \\ \hline x + 2 & 2x^3 - x^2 - 13x - 6 \\ & \underline{2x^3 + 4x^2} \\ & \quad - 5x^2 - 13x \\ & \quad \underline{- 5x^2 - 10x} \\ & \qquad - 3x - 6 \\ & \qquad \underline{- 3x - 6} \\ & \qquad - \; - \; - \end{array}$$

When the linear factor divides exactly into the polynomial it is not necessary to use long division to achieve the initial factorisation, as we saw earlier.

Example 2.10

Given $x + 3$ is a factor of $2x^3 + 3x^2 - 8x + 3$, we may write

$$2x^3 + 3x^2 - 8x + 3 \equiv (x+3)(2x^2 + ax + 1),$$

where a is a constant to be determined. The choice of the quadratic in the second bracket arises because

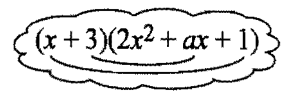
(a) the degree is one less than 3,

(b) $(x + 3)(2x^2 + ax + 1)$ gives the term $2x^3$ which is present on the left hand side,

(c) $(x + 3)(2x^2 + ax + 1)$ gives the term 3 which is present on the left hand side.

The number a is then found by matching the x or x^2 terms on the left hand side with those obtained by multiplying out the brackets.

<u>x term</u>



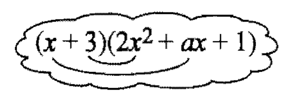
$$-8x \equiv x + 3ax$$

$\therefore \quad -9x \equiv 3ax$

and $\quad a = -3$

or alternatively,

<u>x^2 term</u> $\quad 3x^2 \equiv ax^2 + 6x^2$



so $\quad -3x^2 \equiv ax^2$

or $\quad a = -3$ as before.

Then substitution for this value of a in the initial factorisation gives

$$\begin{aligned} 2x^3 + 3x^2 - 8x + 3 &\equiv (x + 3)(2x^2 - 3x + 1) \\ &\equiv (x + 3)(2x - 1)(x - 1). \end{aligned}$$

The method used in example 2.10 may be used for higher order polynomials.

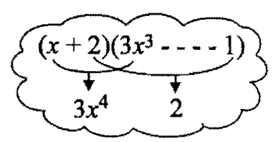
Example 2.11

Write $3x^4 - 8x^2 + 9x + 2$ as a product of a linear factor and a polynomial of degree 3.

We see that $f(-2) = 48 - 32 - 18 + 2 = 0$ so $x + 2$ is a factor.

Then $\quad 3x^4 - 8x^2 + 9x + 2 \equiv (x + 2)(3x^3 + ax^2 + bx + 1),$

where a and b are unknown. As before, the terms $3x^3$ and 1 are easily deduced by matching the terms in $3x^4$ and the constant term. The constants a and b may be found by matching up terms in x, x^2 or x^3.

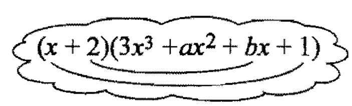


<u>x</u>

$$9x \equiv x + 2bx$$

so $\quad 8x \equiv 2bx$

and $\quad b = 4.$

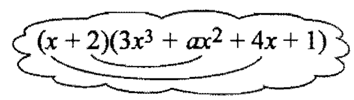


<u>x^2</u>

$$-8x^2 \equiv 4x^2 + 2ax^2$$

so $\quad -12x^2 \equiv 2ax^2$

and $\quad a = -6.$



Substitution of these values for a and b gives

$$3x^4 - 8x^2 + 9x + 2 \equiv (x + 2)(3x^3 - 6x^2 + 4x + 1).$$

Exercises 2.2

1. Find the remainder when the following expressions are divided by the linear expressions indicated.

(i) $x^3 + x - 2$, $x + 1$ (ii) $x^3 - 2x^2 + 3x + 1$, $x - 2$

(iii) $x^4 + x^3 - 2$, $x - 1$ (iv) $x^4 + x^3 - 3x^2 + 1$, $x + 2$

2. If $x^4 + 7x^2 - x + a$ has remainder 2 when divided by $x + 1$, find a.

3. Given that $x^3 + bx^2 + cx - 2$ has remainders 12 and 0 when divided by $x - 2$ and $x + 1$ respectively, find b and c.

4. Factorise the following expressions. Answers may involve quadratic factors.

(i) $x^3 - 3x^2 + 4$ (ii) $x^3 - 2x^2 + 1$

(iii) $x^4 - x^2 + 4x - 4$ (iv) $x^3 - 3x^2 - x + 6$

5. Find the value of k if $x - 2$ is a factor of $x^3 + 6x^2 + kx - 4$.

6. Find the values of a and b if $x^4 - 4x^3 + ax^2 + bx + 24$ is exactly divisible by $x - 2$ and $x + 3$.

7. Given that $x - 2$ is a factor of $f(x) = x^3 + ax^2 - 3x + b$, where a and b are constants, and f has a stationary value when $x = -1$, find the values of a and b. Factorise $f(x)$.

8. Show that no positive integer n can be found such that $x + 4$ is a factor of $x^{2n} + 64$.

9. Find the value of n if $x - 2$ is a factor of $x^{2n} - 64$.

Chapter 3

The binomial expansion for positive integral index

A binomial expression consists of two terms. Thus $2 + x$, $a + b$, $7y + 3$, $x^2 - 7$ and $8a^3 + 2b^2$ are all binomial expressions.
Sometimes it is necessary to multiply out or expand a power of a binomial. For example,

$$\begin{aligned}(a+b)^2 &\equiv (a+b)(a+b) &&\equiv a^2 + ab + ba + b^2 \\ & &&= a^2 + 2ab + b^2, \\ (a+b)^3 &\equiv (a+b)^2(a+b) &&\equiv (a^2 + 2ab + b^2)(a+b) \\ & &&\equiv a^3 + 3a^2b + 3ab^2 + b^3,\end{aligned}$$

after simplifying.
The multiplication process given above is impractical. Fortunately there are two rules for expanding powers: the first uses Pascal's triangle, the second using the concept of combinations.
In this chapter we give only brief consideration to Pascal's triangle and concentrate upon the second method.

3.1 Pascal's triangle

We note that

$$\begin{aligned}(a+b)^0 &= 1, \\ (a+b)^1 &= a + b.\end{aligned}$$

(anything)^0 = 1

Multiplying both sides repeatedly by $(a + b)$, we find after straight forward but tedious calculation that

$$\begin{aligned}(a+b)^2 &= a^2 + 2ab + b^2, \\ (a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3, \\ (a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.\end{aligned}$$

It is possible to write the coefficient of the terms in the above expansions and of others in an arrangement known as Pascal's triangle (we state this without proof).

Binomial expression	Coefficients in the expansion
$(a+b)^0$	1
$(a+b)^1$	1 1
$(a+b)^2$	1 2 1
$(a+b)^3$	1 3 3 1
$(a+b)^4$	1 4 6 4 1
$(a+b)^5$	1 5 10 10 5 1
$(a+b)^6$	1 6 15 20 15 6 1

This arrangement is known as Pascal's triangle and can be continued to as many rows as can be required.

Again, without proof we note that coefficients in any row of the table (excluding the 1's) are found by adding the adjacent coefficients in the preceding row. For example consider the rows for $(a + b)^4$ and $(a + b)^5$, namely

$(a+b)^4$		1		4		6		4		1	
$(a+b)^5$	1		5		10		10		5		1

We give two examples. Thus the 5 in the second row is given by 1 + 4 in the first row; and 10 in the second row is given by 6 + 4 in the first row.
The other numbers 10 and 5 in the second row can be seen to arise in the same way.

Example 3.1
Check the relationship between the coefficients for $(a + b)^5$ and $(a + b)^6$.

	1		5		10		10		5		1	
1		6		15		20		15		6		1

For convenience we check two of the coefficients in row 2 as shown.
The determination of any row of Pascal's triangle enables us to write any binomial expansion.

Example 3.2
Write down the row of coefficients for $(a + b)^6$ and deduce the corresponding row for $(a + b)^7$. Hence expand $(a + b)^7$.

$(a+b)^6$		1		6		15		20		15		6		1	
$(a+b)^7$	1		7		21		35		35		21		7		1

Then

$$(a + b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7.$$

Exercises 3.1

1. Use Pascal's triangle to expand
(i) $(a + b)^5$ (ii) $(a + b)^6$ (iii) $(a + b)^8$.

2. Use Pascal's triangle and write $a = 1$, $b = 2y$ to expand $(1 + 2y)^3$.

3. Expand $(x + 3y)^4$.

4. Expand $(1 - 3y)^3$.

5. Expand and simplify (i) $(1+\sqrt{2})^4$ (ii) $(\sqrt{3}+\sqrt{2})^3$ (iii) $(1+\sqrt{5})^3 + (1-\sqrt{5})^3$.

It is tedious to use Pascal's triangle to expand $(a + b)^{15}$ say, because of the need to generate rows of coefficients. For that reason we introduce another method. To introduce this method we first need to discuss permutations and combinations.

3.2 Permutations

Let's consider the following example.

Example 3.3

Taking the word 'work', how many separate arrangements can be made, taking two letters at a time?

The possible arrangements are conveniently set out as

wo	wr	wk
ow	or	ok
rw	ro	rk
kw	ko	kr

Here, every pair in a row has the same first letter and there are four rows in all: the first letter can be chosen in four ways.

Within a row there are three pairs corresponding to the possible three choices of the second letter (o, r, k in the first row for example).

Then the number of listed arrangements of four letters taken two at a time is

$$12 = 4 \times 3.$$

(4: no. of ways of choosing first letter) (3: no. of ways of choosing second letter)

Definition

Each of the arrangements which can be made by taking all or some of a number of objects is called a **permutation**. In example 3.3 we considered the permutations of four letters taken two at a time.

Suppose now we required the permutations of four letters taken three at a time. Then the first line of the possible arrangements may be taken as

wor wok wro wrk wko wkr

with another three such lines containing o, r, k as their first letters. Then the number of permutations of 4 taken 3 at a time is

$$24 = 4 \times 3 \times 2$$

(4: ways of choosing first letter from four) (3: ways of choosing second from remaining three) (2: ways of choosing third from remaining two)

The general result is that the number of permutations taken r at a time is (*r* factors)

$$n \times (n-1) \times (n-2) \ldots . (n-r+1)$$
$$= n(n-1)(n-2) \ldots . (n-r+1).$$

This number of permutations (n objects taken r at a time) is written ${}^nP_r$.

The number of permutations of n objects taken n at a time is

$${}^nP_n = n(n-1)(n-2) \ldots . (1).$$

For brevity $n(n-1)(n-2) \ldots . (1)$ is written as $n!$

(so, for example, $4! = 4 \times 3 \times 2 \times 1 = 24$).

With this notation, ${}^nP_n = n!$

and $${}^nP_r = n(n-1)(n-2) \ldots . (n-r+1)$$

$$= n(n-1)(n-2)\ldots(n-r+1)\frac{(n-r)(n-r-1)\ldots(1)}{(n-r)(n-r-1)\ldots(1)}$$

$$= \frac{n!}{(n-r)!}.$$

Here we multiply in effect by 1.

Exercises 3.2

1. Write down the number of permutations of 7 objects taken (i) 7 at a time
   (ii) 5 at a time.
2. How many three digit numbers can be made from the set of integers {1,2,3,4,5}?
3. In how many ways can six different books be arranged on a shelf?
4. How many different arrangements can be made from the word 'module', taking 3 letters at a time?
5. Write down the values of $^5P_5$ and $^5P_2$.
6. Evaluate 6! + 2!. **the answer is not 8!**
7. Evaluate $\dfrac{8!}{(4!)^2}$.
8. Evaluate $\dfrac{^7P_4}{4!}$.
9. Write in factorial form $\dfrac{n(n-1)(n-2)}{3\times2\times1}$.
10. Show that $(n-1)! + n! = (n+1)[(n-1)!]$.

3.3 Combinations

Let's consider again the permutations of the letters of 'work', taken 3 at a time. We saw that the number of permutations is

$$^4P_3 = 4\times3\times2 = 24.$$

Suppose now we wish to know how many <u>sets</u> of 3 can be taken from the letters of 'work', counting one set once.
The number of sets is 4 :-

wor, wok, wrk, ork

Note the letters wor do not also occur as wro, owr, rwo, for example.

We refer to the number of sets of 4 objects taken 3 at a time as the number of **combinations** of four objects taken 3 at a time.

Definition

The number of combinations of n objects taken r at a time is the number of sets (order being ignored) of n objects taken r at a time. It is denoted by the symbol $\binom{n}{r}$. Then from the above situation, $\binom{4}{3} = 4$.

Example 3.4

Find the combination $\binom{3}{2}$ of A, B, C taking two letters at a time.

The combinations are AB, BC, CA

$$\text{so} \quad \binom{3}{2} = 3.$$

It is clear that only limited progress is possible if we have to write down all possible combinations; for example the UK lottery relates to selection of 6 numbers from 49 and $\binom{49}{6}$ = 13, 983, 816, nearly 14 million.

Even if you could write down a selection every 5 seconds, it would take over 2 years to list all the cases.

As it happens, a convenient formula for $\binom{n}{r}$ exists.

To derive this formula we first consider an example.

Example 3.5

Consider the following method of finding ${}^5P_3$, the number of permutations of 5 objects A, B, C, D, E (say), taken 3 at a time. We know this is $\frac{5!}{2!} = 60$, but never mind.

First, we consider the selection of 3 objects taken from A, B, C, D, E, counting a particular set of 3 just once, e.g. the set A, B, C is counted once and the particular order of selection is unimportant. The number of such sets is $\binom{5}{3}$, by definition.

Now, for the permutation situation where order is important, each set of 3 such as A, B, C will generate six permutations (ABC, ACB, BAC, BCA, CAB, CBA). Thus each set of 3 objects will generate ${}^3P_3 = 3! = 3 \times 2 \times 1$ permutations.

Then we may regard the total number of permutations of 5 taken 3 at a time as being generated by first taking sets of 3 and then rearranging each set of 3 in ${}^3P_3$ ways. Thus,

$${}^5P_3 = \binom{5}{3} \times 3!.$$

total number of permutations of 5 taken 3 at a time = **number of combinations of 5 taken 3 at a time** **number of permutations of a set of 3 taken 3 at a time**

so $\binom{5}{3} = \frac{{}^5P_3}{3!} = \frac{5!}{2!3!}$, on substituting for ${}^5P_3$.

$\frac{5!}{2!3!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 3 \times 2 \times 1} = 10$

More generally,

$${}^nP_r = \binom{n}{r} \times r!$$

so $$\binom{n}{r} = \frac{{}^nP_r}{r!}.$$

$$\therefore \quad \binom{n}{r} = \frac{n!}{(n-r)!r!}.$$

${}^nP_r = \frac{n!}{(n-r)!}$

Example 3.6
Out of 15 men, in how many ways can 11 be chosen? If the men were to form a cricket team, find the total number of batting orders that are possible.

$$\text{Number of sets of 11 from 15} = \binom{15}{11}$$
$$= \frac{15!}{4!11!} = \frac{15 \times 14 \times 13 \times 12 \times 11!}{4 \times 3 \times 2 \times 1 \times 11!}$$
$$= 1365.$$

Note the convenience of using 11! in expression for 15!

The total number of batting orders is the number of permutations of 15 taken 11 at a time $= {}^{15}P_{11} = \frac{15!}{4!} = 54,486,432,000$ – a lot of batting orders.

Exercises 3.3

1. Find the values of (i) ${}^9P_8$ (ii) ${}^{26}P_6$ (iii) $\binom{25}{5}$ (iv) $\binom{20}{15}$

2. How many different arrangements can be made by taking six of the letters of the word equations?

3. How many different numbers can be made by selecting 4 digits from the set $\{2, 3, 5, 6, 7, 8\}$? How many start with 8?

4. Find the possible values of n if
$$\binom{n}{3} = 10\binom{n}{5} \quad \text{given that } n > 4.$$

5. A team of 4 senior citizens is to be selected from a group of 20 to compete in a national quiz. In how many ways can the team be chosen if
(a) any four can be chosen,
(b) the four chosen must include the 'resident brain'?

6. In the U.K. the national lottery started in 1995, 6 balls being selected from 49. In how many ways can a set of 5 be selected, one set being counted just once.

3.4 The Binomial Theorem

In this section the expansion of $(x+a)^n$ is considered, where n is a positive integer. We start by considering the expansions for $(x + a)(x + b)$, $(x + a)(x + b)(x + c)$ and $(x + a)(x + b)(x + c)(x + d)$.

Now $\quad (x + a)(x + b) = x^2 + (a + b)x + ab$

and $\quad (x + a)(x + b)(x + c) = [x^2 + (a + b)x + ab][x + c]$

$$= x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc.$$

Similarly,

$$(x + a)(x + b)(x + c)(x + d) = x^4 + (a + b + c + d)x^3 + (ab+bc+ad+bd+cd+ac)x^2 + (abc + abd + bcd + acd)x + abcd.$$

Focusing on this last result, we see that the coefficients are as follows :-

x^4 1

x^3 sum of the letters one at a time i.e. $(a + b + c + d)$,

x^2 sum of the products of the letters two at a time i.e. $(ab + bc + ad + bd + cd + ac)$,

x sum of the products of the letters 3 at a time, i.e. $(abc + abd + bcd + acd)$, and the term independent of x is $abcd$.

Now we know that the number of ways of

(i) grouping 4 letters 1 at a time is $\binom{4}{1} = \frac{4!}{1!3!} = 4$,

(ii) grouping 4 letters 2 at a time is $\binom{4}{2} = \frac{4!}{2!2!} = 6$,

(iii) grouping 4 letters 3 at a time is $\binom{4}{3} = \frac{4!}{3!1!} = 4$,

(iv) grouping 4 letters 4 at a time is $\binom{4}{4} = 1$.

In the preceding, let $b = c = d = a$. Then

$$(x + a)^4 = x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4$$

or $$(x + a)^4 = x^4 + \binom{4}{1}ax^3 + \binom{4}{2}a^2x^2 + \binom{4}{3}a^3x + a^4.$$

By a similar process we could obtain

$$(x + a)^5 = x^5 + \binom{5}{1}ax^4 + \binom{5}{2}a^2x^3 + \binom{5}{3}a^3x^2 + \binom{5}{4}a^4x + a^5.$$

From a consideration of these general results we may conjecture but not prove the general results for any positive integer n :-

$$(x + a)^n = x^n + \binom{n}{1}ax^{n-1} + \binom{n}{2}a^2x^{n-2} + \binom{n}{3}a^3x^{n-3} + \ldots + a^n.$$

A more convenient form of the expansion when the various $\binom{n}{1}$, $\binom{n}{2}$ etc. are written out in full is available :-

The binomial theorem for positive integer *n*

$$(x+a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{1.2}a^2x^{n-2} + \frac{n(n-1)(n-2)}{1.2.3}a^3x^{n-3} + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}a^4x^{n-4} + \ldots + a^n.$$

Example 3.7

Expand $(x + a)^6$.

In this $n = 6$ and

$$(x+a)^6 = x^6 + 6ax^5 + \frac{6(6-1)}{1.2}a^2x^4 + \frac{6(6-1)(6-2)}{1.2.3}a^3x^3 + \frac{6(6-1)(6-2)(6-3)}{1.2.3.4}a^4x^2 + \frac{6(6-1)(6-2)(6-3)(6-4)}{1.2.3.4.5}a^5x + a^6$$

$$= x^6 + 6ax^5 + 15a^2x^4 + 20a^3x^3 + 15a^4x^2 + 6a^5x + a^6.$$

Example 3.8

Expand $(2x + 3y)^4$.

We write $a = 2x$, $b = 3y$ in the expansion for $(a + b)^4$. Then

$$(2x+3y)^4 = (2x)^4 + 4(2x)^3(3y) + \frac{4.3}{1.2}(2x)^2(3y)^2 + \frac{4.3.2}{1.2.3}(2x)(3y)^3 + (3y)^4$$

$$= 16x^4 + 96x^3y + 216x^2y^2 + 216xy^3 + 81y^4.$$

Example 3.9

Expand $(a - 2x)^3$.

Note the – sign

Here we write $b = -2x$. Then

$$(a-2x)^3 = a^3 + 3a^2(-2x) + \frac{3.2}{1.2}a(-2x)^2 + (-2x)^3$$

$$= a^3 - 6a^2x + 12ax^2 - 8x^3.$$

Example 3.10

Find the term in x^2 in the expansion of $\left(x+\frac{1}{x}\right)^6$.

It is convenient to write

$$\left(x+\frac{1}{x}\right)^6 = x^6\left(1+\frac{1}{x^2}\right)^6$$

and look for the term in $\frac{1}{x^4}$ in the expansion of $\left(1+\frac{1}{x^2}\right)^6$.

Now $$\left(1+\frac{1}{x^2}\right)^6 = 1 + \frac{6}{x^2} + \frac{6.5}{1.2}\left(\frac{1}{x^2}\right)^2 + \ldots$$

$(a+b)^6$ with $a = 1$, $b = 1/x^2$

The term in $\frac{1}{x^4}$ is $15.\frac{1}{x^4}$.

Thus the term in x^2 in $\left(x+\frac{1}{x}\right)^6$ is

$$x^6 \cdot \frac{15}{x^4} = 15x^2.$$

Example 3.11

If the coefficients of the 3rd and 5th terms in the expansion of $(1 + x)^n$ are equal, find n given that $n > 3$.

The 3rd term is the term in x^2 and is $\frac{n(n-1)}{1.2}x^2$. The 5th term is the term in x^4 and is $\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}x^4$.

Then $$\frac{n(n-1)}{1.2} = \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}.$$ (coefficients are equal)

This can be cancelled down to

$$1 = \frac{(n-2)(n-3)}{3.4} \quad \text{since } n \neq 0,1.$$ ($n > 3$)

$\therefore \quad 12 = n^2 - 5n + 6$

so $\quad n^2 - 5n - 6 = 0.$

Factorise: $(n+1)(n-6) = 0.$

Then $n + 1 = 0$ so $n = -1$ (impossible)

or $\quad n - 6 = 0$ so $n = 6$ which is the solution.

Exercises 3.4

1. Expand (i) $(1 + 2z)^5$ (ii) $(x - 2y)^4$ (iii) $\left(x+\frac{1}{x}\right)^3$ (iv) $(2y - z)^3$
2. Write down the first three terms of the expansions of
   (i) $(1 + x)^{12}$ (ii) $(1 - 2y)^{14}$ (iii) $(p + q)^{16}$
   (iv) $\left(1+\frac{x}{2}\right)^{10}$ (v) $(2 - 3x)^8$ (vi) $\left(x^2+\frac{1}{x^2}\right)^{11}$
3. In the expansion of $(2x - y)^{20}$ find the term containing y^3.
4. In the expansion of $(1 - 2x)^{10}$ find the term containing x^3.
5. By substituting $x = 0.01$ in the binomial expansion of $(1 - 2x)^8$, find $(0.98)^8$ correct to four decimal places.
6. By substituting 0.1 for x in the binomial expansion of $\left(1+\frac{x}{10}\right)^8$, find the value of $(1.01)^8$ correct to four significant figures.
7. If x is so small that x^3 and higher powers are negligible, show that $(3 - 2x)(1 + 2x)^{10} \approx 3 + 58x + 500x^2$.

8. Show that if x is small enough for x^2 and higher powers of x to be neglected, the function $(x + 2)(1 - 3x)^8 \approx 2 - 47x$.

9. Find y if $(1 - 3y)^3 + (1 + 3y)^3 = 218$.

10. Write down the coefficients of x^4 and x^5 in the binomial expansion of $(1 + ax)^{10}$. Given that the first coefficient above is 8 times the second, find the value of a.

11. In the binomial expansion of $(3 + x)^n$, the coefficient of x^2 is 1.5 times the coefficient of x^3. Find the value of n.

12. In the binomial expansion of $(a + x)^8$, the coefficient of x^3 is 28 times the coefficient of x. Find the value of a.

Chapter 4

Functions

In **P1** we discussed briefly the concept of functions, specifically polynomial functions. We described as functions expressions in x which take values in response to the allocation of values of x. Thus for example,

if $\quad f(x) = x^2 + 2x - 3$

then $\quad f(-1) = (-1)^2 + 2(-1) - 3 = -4,$

$\quad f(0) = (0)^2 + 2(0) - 3 = -3,$

and so on.

In this chapter we develop further the idea of function. One way in which we develop the idea is to allow expressions other than polynomials.

Example 4.1

$$f(x) = x + \frac{1}{x},$$
$$g(x) = \sqrt{x+1}$$

are two possible functions for consideration. Whilst f is often used in functions, we also use g and other letters to denote other functions.

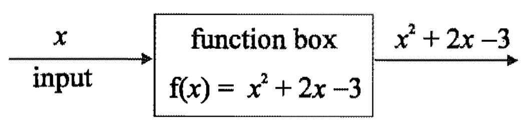
4.1 Functions and processes

Let's look again at the idea of function by means of a block diagram.

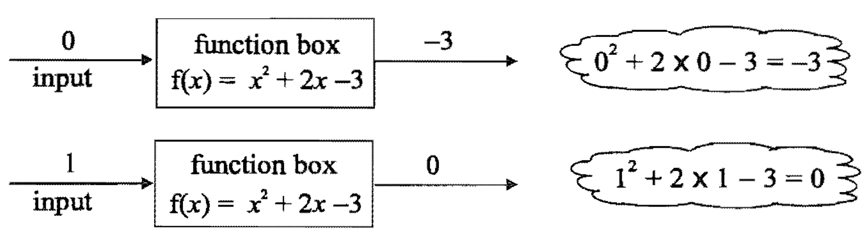
Example 4.2

Consider $f(x) = x^2 + 2x - 3$.

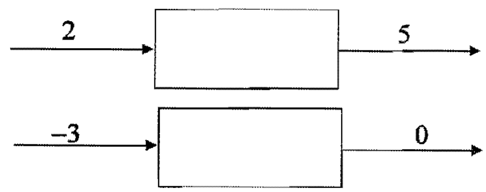
This can be represented as follows.



The function box is considered to be a black box, i.e. a device which by some means takes an input and generates an output. Then for various inputs :-



Other inputs and outputs are



We have introduced the black box to underline the point that a function is essentially a process for generating outputs from given inputs. It should be noted that the process (as opposed to the output) doesn't depend upon the letter used.

Thus $\quad f(x) = x^2 + 2x - 3$

and $\quad f(a) = a^2 + 2a - 3$

are essentially the same process :- given a number, square it, add twice the number, and then subtract 3.

Example 4.3

Write down two further representations of the process

$$g(x) = 3x + 4.$$

Any other letters may be used :-

$$g(a) = 3a + 4,$$
$$g(b) = 3b + 4.$$

A second point to be noted is that given an input, then the output is uniquely defined. Thus

$$f(x) = \sqrt{x+4} \quad \text{is a function.}$$

However

$$g(x) = \pm\sqrt{x+4}$$

is not a function because two possible answers or outputs could be obtained from one input, e.g.

$$g(5) = \pm\sqrt{5+4} = \pm 3.$$

Rule

A function must give one answer for any one "acceptable" input.

Don't worry about the word acceptable at this stage.

In passing it should be noted that the same output may be obtained with different inputs. In example 4.2 with

$$f(x) = x^2 + 2x - 3$$

we see that $\quad f(1) = 0$

and $\quad f(-3) = 0,$

i.e. the output 0 can arise with two different inputs.

Exercises 4.1

1. If $f(x) = x + \dfrac{1}{x}$, write down f(1), f(2), f(−1) and f(a).

2. Is $g(x) = \dfrac{1}{x^2} \pm x$ a possible function process?

3. Show that for $h(x) = (x+2)^2 + 3$, $h(0) = h(-4)$.

4. Are there any values of x which would be unacceptable inputs for $f(x) = \dfrac{2}{(x+1)(x+2)}$? Hint : Can you find $\dfrac{2}{0}$? Try it on your calculator.

4.2 Domains of functions

In section 4.1 we regarded the function expression as a process to generate outputs from given inputs, the rule being that the input should generate one output. Here we consider the acceptability, or otherwise, of inputs.

Example 4.4

Consider $f(x) = \sqrt{x}$ and $g(x) = \dfrac{1}{x}$.

Are we able to find f(−2) and g(0)? Try it on your calculator!
In fact, it is impossible to find outputs for these particular inputs and processes.

Example 4.4 indicates that it is often useful to specify, along with the rule, formula or process, the elements upon which the process acts. Thus the $\sqrt{x}$ rule will only act upon values of x which are non-negative; and the $\dfrac{1}{x}$ rule will only act upon non-zero values of x, i.e. when $x \neq 0$.
The set of numbers upon which a rule or process is able to act is called the **domain**.

Example 4.5

What is the largest possible domain for

(a) $f(x) = \dfrac{1}{x+1}$, (b) $g(x) = (x+1)^2$, (c) $h(x) = \dfrac{1}{\sqrt{x+2}}$?

We adopt the convention that, unless otherwise stated, the process or rule will act upon all real numbers with possible certain exceptions which are always noted.

Thus (a) $f(x) = \dfrac{1}{x+1}$, $(x \neq -1)$

(b) $g(x) = (x+1)^2$, (no exceptions, i.e. all values of x are allowed)

(c) $h(x) = \dfrac{1}{\sqrt{x+2}}$. $(x > -2)$

In interval notation the domains are

(a) $(-\infty, -1) \cup (-1, \infty)$ (b) $(-\infty, \infty)$ (c) $(-2, \infty)$.

Note that in (c), $x = -2$ is not allowed because division by 0 is not defined.

Sometimes even when the rule or process would accept all or most numbers, we may wish to restrict the domain.

Example 4.6

$$f(x) = x^2 + 4x + 5, \qquad 0 \le x \le 5$$
$$g(x) = x^3 + 4. \qquad 4 \le x < 16$$

The domains here are therefore [0, 5] and [4, 16).

Exercises 4.2

1. Find the largest possible domain in interval notation of the functions defined by the following rules :-

(a) $f(x) = \sqrt{2-x}$ (b) $f(x) = \dfrac{x-2}{x+3}$ (c) $f(x) = \dfrac{1}{\sqrt{x}}$

(d) $f(x) = \dfrac{1}{\sqrt{x+2}} + \dfrac{1}{x-1}$ (e) $f(x) = \sqrt{(x+1)(x-1)}$

(f) $f(x) = \dfrac{2}{(x-1)(x+3)}$ (g) $f(x) = \sqrt{\dfrac{2x-1}{x+3}}$

(h) $f(x) = \sqrt{4-x^2}$ (i) $f(x) = \sqrt{\dfrac{9-x}{16-x}}$

4.3 Ranges of functions

In the last section we considered the domain of a function, the set of numbers upon which the rule acts. We now consider the set of numbers which are produced by the process, i.e. the outputs. The set of outputs is called the **range** of the function.

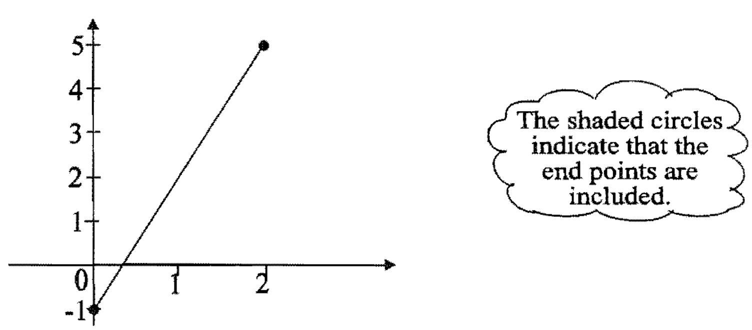
When asked to find the range by the action of a given function rule upon a domain, we often find it useful to use a graphical representation of functions.

Example 4.7

Find the ranges in interval notation when the given function rule acts on the elements of the given domain :-

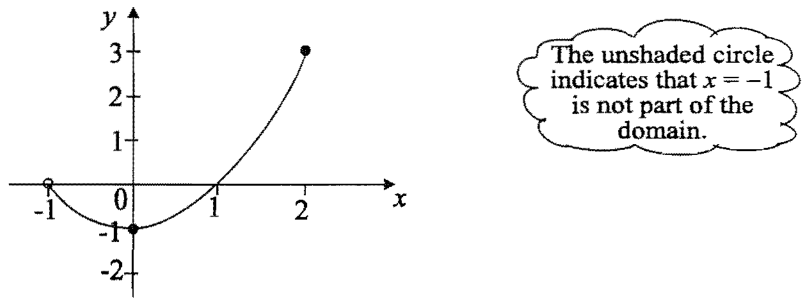
(i)	$f(x) = 3x - 1$	[0, 2]
(ii)	$f(x) = x^2 - 1$	(−1, 2]
(iii)	$f(x) = x^2 + 2x + 3$	(−∞, ∞)

(i) We write $f(x) = 3x - 1$ as $y = 3x - 1$.
Then y is the output for a particular x. The introduction of y allows us to plot inputs (x) and outputs (y) on a graph in the usual way. In this case the graph is a straight line as shown.



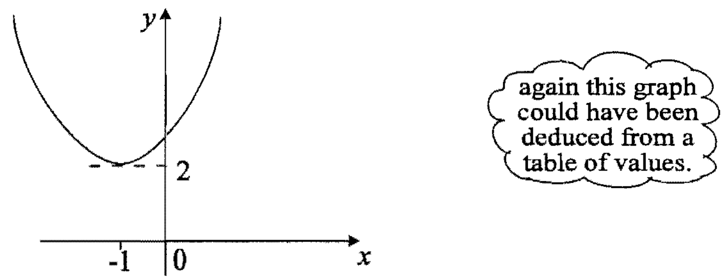
The range is [−1, 5].

(ii) Here $y = x^2 - 1$ and for the given domain $(-1, 2]$, it is easy (by means of a table, for instance) to draw the graph.



Here the range is $(-1, 3]$, the endpoints of this interval being the lowest and highest points of the graph. It should be noted that finding the outputs for the endpoints of the domain would have resulted in $(0, 3]$ which is not the range in this case.

(iii) In this case $y = x^2 + 2x + 3 = (x + 1)^2 + 2$, on completing the square. The smallest value arises when $x = -1$, this smallest value being 2. The range is then $[2, \infty)$.



Exercises 4.3

1. Draw the graphs and state the ranges in the following cases :-

(a) $f(x) = 1 - x$, domain $[-3, 2]$ (b) $f(x) = (x - 2)^2 + 4$, domain $(0, 4)$

(c) $f(x) = \sqrt{x+2}$, domain $(-1, 7]$ (d) $f(x) = \dfrac{1}{x^2}$, $(x > 0)$

(e) $f(x) = x^2$, domain $[-4, 5]$.

2. For each of the following function rules and domains, decide whether there are two or more elements in the domain corresponding to a single element of the range, [for example for $f(x) = x^2$, all x, $f(-2) = f(2) = 4$ so -2 and 2 in the domain correspond to 4 in the range].

(a) $f(x) = 3x + 4$ $[-1, 20]$ (b) $f(x) = x^2 + 8x$ $[-4, 4]$
(Note that $x^2 + 8x \equiv (x + 4)^2 - 16$.)

(c) $f(x) = x^2 + 8x$ $[-5, 1]$ (d) $f(x) = \dfrac{1}{x}$ $(x > 0)$

(e) $f(x) = \dfrac{1}{x^2} \quad (x > 0)$ (f) $f(x) = \dfrac{1}{x^2 + 2} \quad (-\infty, \infty)$

(g) $f(x) = 4x - x^2 \quad [0, 4]$

$4x - x^2 \equiv 4 - (x-2)^2$

Summary

A function involves three components, namely :-

1. a rule or formula (regarded as a process) which gives a single output value for a single input value;
2. a set of input values upon which the rule acts, known as the *domain*;
3. a set of output values, known as the *range*.

4.4 Inverse functions

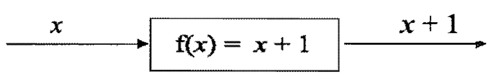
Given a function f, involving the three components mentioned earlier, it is sometimes possible to find another function which reverses the effect of f. This function would take an output of f and find the input from which it came. Such a new function is called an **inverse function** and may or may not exist. Here we consider which functions have inverse functions and show how such inverse functions may be found.

Example 4.8

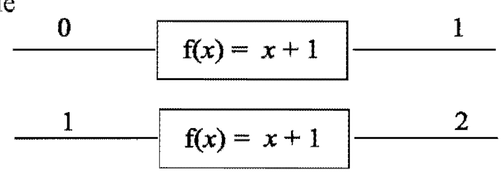
Given $f(x) = x + 1 \quad (-\infty, \infty)$,

can we find a function which reverses the effect of f?

The process could be represented as a black box.

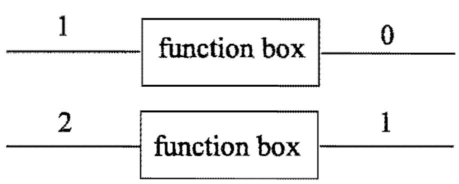


So for example



etc.

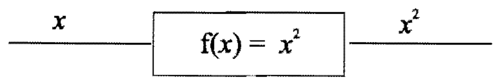
Given an output, are we able to say what the input was? A little thought shows that for this case, a given output arises from an input which is one less. Thus the reverse process could be represented as



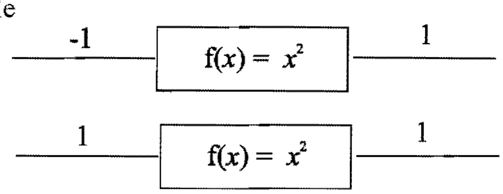
We consider how to find the new rule later.

It is clear that a function exists which reverses $f(x) = x + 1$: it is the function which subtracts 1 from the input value.

Example 4.9
Given $f(x) = x^2 \quad (-\infty, \infty)$,
can we find a function which reverses the effect of f?
The process could be represented as a black box.



So for example



Given an output, are we able to say what the input was? A little thought shows that for an output there may be two inputs, for example in the above, an output of 1 could arise from inputs -1 and 1.
In this case we are unable to find a function which reverses the effect of $f(x) = x^2$.
Let's summarise the results of Examples 4.8 and 4.9.

Function		Does the inverse function exist?
$f(x) = x + 1$	An output arises from one input only	Yes
$f(x) = x^2$	There are two inputs giving the same output	No

Functions such as $f(x) = x + 1$ (any domain), for which only one input gives a particular output, are said to be **one-one** functions. Note that $f(x) = x^2$ with domain $(-\infty, \infty)$ is not a one-one function (as seen earlier). One-one functions arise again in Chapter 12.
Examples 4.8 and 4.9 illustrate (but do not prove) a general result concerning one-one functions and inverse functions. First, we recall the definition of an inverse function.

Definition
The inverse function f^{-1} of a function f is a function which takes the outputs of f and maps them to the inputs of f, in other words f^{-1} reverses the effect of f.

Rule
A function f has an inverse function f^{-1} only if f is a one-one function.

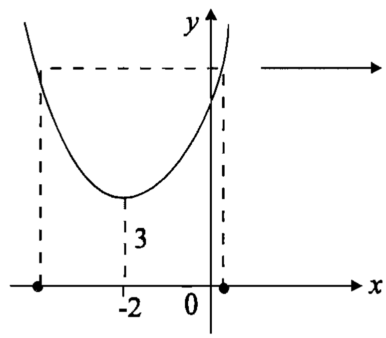
Example 4.10
Show that for $f(x) = x^2 + 4x + 7$ with domain $(-\infty, \infty)$, no inverse function exists.
Suppose we have an output y and attempt to find the corresponding input x.
Thus let $y = x^2 + 4x + 7$ and attempt to find x in terms of y.

$\therefore \quad x^2 + 4x + 7 - y \;=\; 0.$

By the quadratic formula,

$$x \;=\; \frac{-4 \pm \sqrt{16 - 4(7-y)}}{2}$$
$$\;=\; \frac{-4 \pm 2\sqrt{4 - 7 + y}}{2} \;=\; -2 \pm \sqrt{y-3}.$$

For real roots, $y \geq 3$ so that the range of f is $[3, \infty)$. For a given y in this range, there are two values of x. This is illustrated in the graph of $y \;=\; x^2 + 4x + 7$.

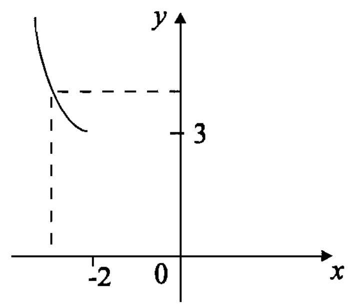


Two values correspond to this value of y.

In this case, therefore, the inverse function doesn't exist.

Example 4.11

We note that if the function had been $f(x) \;=\; x^2 + 4x + 7$ (same rule as in 4.10) with a restricted domain $(-\infty, -2]$ the graph would be as shown, and there is only one input of a given y, where $y \geq 3$.



The range of f is $[3, \infty)$.

Then the inverse function exists and for given y,

$$x \;=\; -2 - \sqrt{y-3}$$

from example 4.10.

Thus the function f^{-1} processes y to give

$$-2 - \sqrt{y-3}.$$

In other words,

$$f^{-1}(y) \;=\; -2 - \sqrt{y-3}.$$

The –ve square root is taken because all x's are –2 or less.

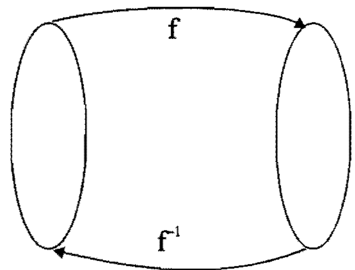
The two square roots are not possible, i.e. f is one - one.

Now any letter may be used in the definition of the rule for f^{-1} and usually we choose the letter to be x. Then in this case

$$f^{-1}(x) \;=\; -2 - \sqrt{x-3}$$

is the inverse of $f(x) = x^2 + 4x + 7$ when the domain of f is $(-\infty, -2]$.

Given the rule of any function, it is appropriate to state also the domain and range of that function. To determine those entities for f^{-1}, we note the following diagram, where we have represented the domain and range of f as two boxes.



Now an examination of the diagram shows that f^{-1} takes elements of the right box (i.e. the range of f) and finds corresponding elements in the left hand box (the domain of f).
In other words,

$$\text{range of f} = \text{domain of } f^{-1},$$
$$\text{domain of f} = \text{range of } f^{-1}.$$

This is, in fact, the relationship between domains and ranges of one-one functions and their inverse functions. In the present case, therefore,

<u>f</u>	<u>f^{-1}</u>
Domain $(-\infty, -2]$	Domain $[3, \infty)$
Range $[3, \infty)$	Range $(-\infty, -2]$

Summary

(i) When a function f is one-one, its inverse f^{-1} exists.
(ii) The domain of f^{-1} = the range of f.
(iii) The range of f^{-1} = the domain of f.
(iv) To find the rule for f^{-1} we
(a) attempt to solve $y = f(x)$ for x in terms of y so that $x = f^{-1}(y)$.
(b) replace y by x to give $f^{-1}(x)$ in the usual notation.
Note that if two or more values result in (a) it is possible that the original function f is not one-one over the given domain (Example 4.10); but it may be one-one over a particular restricted domain (Example 4.11).

Example 4.12

Given $\quad f(x) = \sqrt{\dfrac{3-x}{x-5}},$

state the largest possible domain and the corresponding range. Find $f^{-1}(x)$ and state the domain and range for f^{-1}.

Since f(x) involves a square root, we must choose x so that the square root of a non-negative quantity is being taken, i.e. choose x so that

$$\frac{3-x}{x-5} \geq 0.$$

Note $x = 5$ is not allowed

then we require $3 - x \geq 0$ and $x - 5 > 0$ (i)
or $3 - x \leq 0$ and $x - 5 < 0$ (ii)
(i) leads to $x \leq 3$ and $x > 5$ which cannot occur together.
(ii) leads to $x \geq 3$ and $x < 5$ which can be combined into the one statement $3 \leq x < 5$, so that the largest possible domain of f(x) is [3, 5).
For the corresponding range, we note that when $x = 3$, f(x) = 0 and as $x \to 5$, f(x) $\to \infty$. Hence the range of f is $[0, \infty)$.
To find $f^{-1}(x)$, let

$$y = f(x) = \sqrt{\frac{3-x}{x-5}}.$$

Then $$y^2 = \frac{3-x}{x-5}$$

so that $$(x-5)y^2 = 3 - x$$

and $$x(y^2+1) = 3 + 5y^2.$$

$$\therefore \quad x = \frac{3+5y^2}{y^2+1}$$

Only one value of x for a given value of y, i.e. f is one - one.

so that $$f^{-1}(y) = \frac{3+5y^2}{y^2+1}.$$

Changing from y to x, we have

$$f^{-1}(x) = \frac{3+5x^2}{x^2+1}.$$

The domain of f^{-1} = range of f which is $[0, \infty)$ and the range of f^{-1} = domain of f which is [3,5).

Exercises 4.4

1. For each of the following functions, by drawing rough graphs or otherwise, decide whether f is a one-one function :-

(a) $f(x) = 4x + 3$ domain $[-1,5]$
(b) $f(x) = 2x^2 + 1$ domain $[-3,3]$
(c) $f(x) = x^2 + 6x$ domain $[-3, 3]$
(d) $f(x) = x^2 + 6x$ domain $[-5,5]$
(e) $f(x) = \sqrt{x^2+4}$ domain $(-\infty, \infty)$
(f) $f(x) = (x+2)^2$ domain $(-\infty, \infty)$
(g) $f(x) = \frac{1}{x}$ domain $(0, \infty)$
(h) $f(x) = \frac{1}{x^2+2x+3}$ domain $(-\infty, \infty)$

2. For all functions which are one-one in question 1, find their inverse functions (give the three components).

3. Find the inverses of the following one-one functions, stating the rule, domain and range in each case :-

(a) $f(x) = 1 + \frac{1}{x}$ domain $(0, \infty)$

(b) $f(x) = (x+2)^2 + 3$ domain $(0, \infty)$

(c) $f(x) = (x+1)(x+5)$ domain $(-3, \infty)$

(d) $f(x) = \sqrt{x+2}$ domain $[-2, \infty)$

(e) $f(x) = \frac{1}{\sqrt{x+2}}$ domain $(-2, \infty)$

(f) $f(x) = \sqrt{(x+3)(x-1)}$ domain $[1, \infty)$

4. Find the largest possible domain for the function $f(x) = \frac{1}{x+1}$, and find the inverse of f, stating the domain and range of f^{-1}.

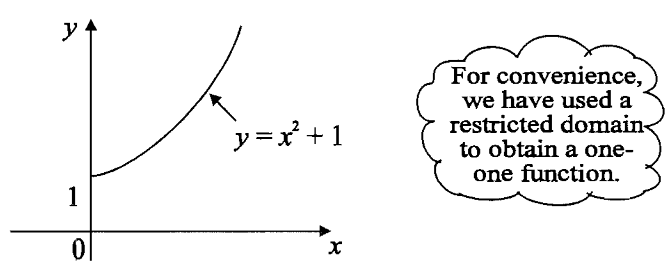
5. Given $f(x) = \frac{1}{x^2+1}$ with domain $(-\infty, 0)$, find f^{-1}, stating the domain and range.

6. Find the largest possible domain of $f(x) = \frac{2x-1}{x+3}$, giving the corresponding range. Find $f^{-1}(x)$.

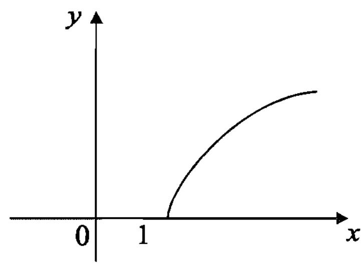
7. Allowing non-negative values only, find the greatest possible domain for the function f given by $f(x) = \sqrt{4-x^2}$. Find (i) the range of f,
(ii) the rule for f^{-1}, i.e. $f^{-1}(x)$.

4.5 Sketching inverse functions

In section 4.3 we demonstrated how a function can be represented as a graph. For example, if $f(x) = x^2+1$ with domain $[0, \infty)$ we write $y = x^2+1$ and obtain the graph

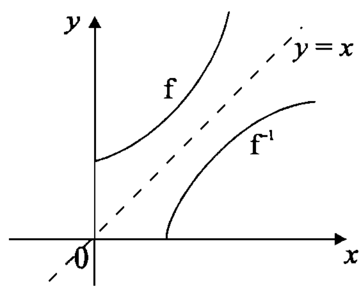


This function has range $[1, \infty)$. It is easy to show that the inverse function is $f^{-1}(x) = \sqrt{x^2-1}$ with domain $[1, \infty)$ and range $[0, \infty)$.
To draw this graph of f^{-1} we adjust the graph of f so that the input axis is horizontal and the output axis is vertical. Then the graph of f^{-1} is



> **The new axes 0x, 0y are the previous 0y, 0x axes respectively.**

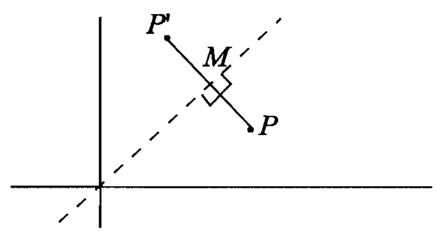
In fact, the graph of f^{-1} is easily obtained by reflecting the graph of f in the line $y = x$.



> **If the line $y = x$ is a mirror and pins are placed upright on the graph of f we'll see pins behind the mirror situated on the graph of f^{-1}**

Technique

To find the reflection P' of any point P in the line $y = x$ we drop a perpendicular PM from P to the line $y = x$ and continue PM to P' where $MP' = PM$. P' is then the reflection of P in the line.



Rule

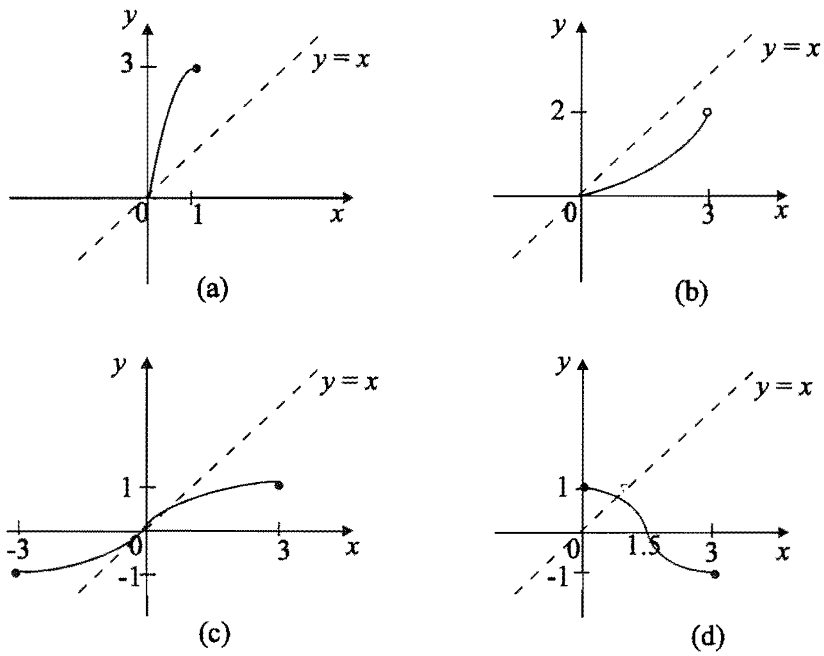
The graph of f^{-1}can be found from the graph of f by reflecting the latter in the line $y = x$.

Exercises 4.5

1. Given $f(x) = 3x + 2$, (all x), find f^{-1}, and draw the graphs of f and f^{-1} on the same diagram.

2. Given $f(x) = x^2$ $[0, \infty)$, find f^{-1}. Draw the graph of f^{-1} by reflecting the graph of f in the line $y = x$.

3. Given the following graphs of functions, draw the graphs of the corresponding inverse functions.



4.6 Composition of functions

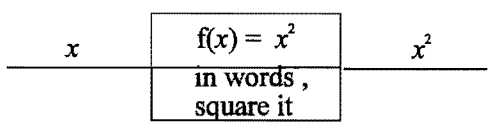
In this section we discuss a method of combining functions to obtain new functions. Let's consider the following example.

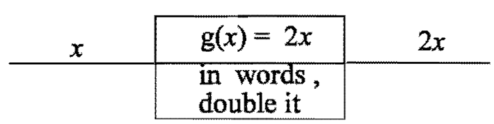
Example 4.13

Let f(x) $= x^2$ with domain $(-\infty, \infty)$,

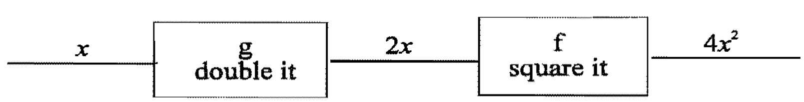
g(x) $= 2x$ with domain $(-\infty, \infty)$.

In the block diagram representation:-



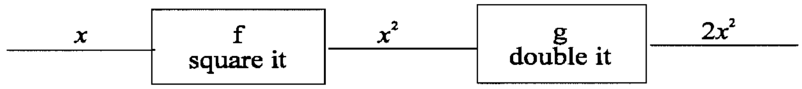


Let's consider the following function which is a combination of f and g.



If we call this new function or <u>composite</u> function h then h(x) $= 4x^2$.

If the f box preceded the g box as shown, the input x would result in an output of $2x^2$.



Thus if we call the effect of combining f and g here a function k then $k(x) = 2x^2$.

Definition
The result of performing a function f *first* and *then* g is a function gf where $gf(x) = g(f(x))$.
The order should be noted : gf means that the function f is performed first.

> **If $f(x) = x^2$, $g(x) = 2x$, $gf(x) = g(x^2) = 2x^2$.**

Similarly, the function resulting from performing g *first* and *then* f is a function fg where $fg(x) = f(g(x))$.
The process of combining functions in this way is called **composition of functions**.
It is not essential to draw the block diagrams when carrying out composition of functions.

Example 4.14
If $f(x) = 3x - 2$
$g(x) = x^2 + 1$ (domains $-\infty, \infty$)
find $fg(x)$ and $gf(x)$.

> **Remember that the letter x used in the definition of the functions is unimportant. For instance g means 'square it and add one'.**

Now $fg(x)$ means $f(g(x))$ which we find by replacing x by $g(x)$ in the expression for $f(x)$.

$$f(g(x)) = f(x^2 + 1) = 3(x^2 + 1) - 2$$
$$= 3x^2 + 3 - 2 = 3x^2 + 1.$$

> **f means 'multiply it by 3 and then subtract 2'.**

Similarly, $gf(x)$ means $g(f(x))$ which we find by replacing x by $f(x)$ in the expression for $g(x)$.

$$g(f(x)) = g(3x - 2) = (3x - 2)^2 + 1$$
$$= 9x^2 - 12x + 5.$$

> **g means 'square it and add 1'**

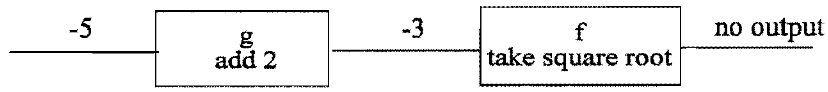
It is tempting to assume that given two functions f and g we may always be able to form the functions fg and gf. This is not the case : sometimes we are unable to form one or the other.

Example 4.15
Consider the functions

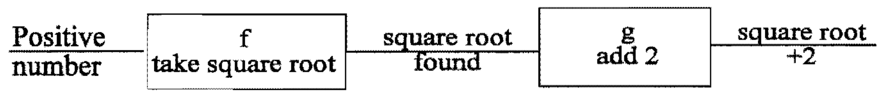
$f(x) = \sqrt{x}$, domain $(0, \infty)$
$g(x) = x + 2$. domain $(-\infty, \infty)$
Then $fg(x) = f(g(x)) = \sqrt{x+2}$,
and $gf(x) = g(f(x)) = \sqrt{x} + 2$.

We note however that
$f(g(x)) = \sqrt{x+2}$ is not defined for $x < -2$ and such values of x are allowed for in the domain of g, the first function to act. This may be better

understood by referring to the block diagram and considering the processing of −5, for example.



In contrast, there is no difficulty in forming gf(x), where the function f acts first. The domain for f is (0, ∞), i.e. the set of non-negative numbers and the square root process is able to deal with such numbers.



An obvious question poses itself here : in general terms, why does fg exist and gf not exist?
A little thought leads to the following conclusion : gf doesn't exist because some of the outputs of the first function (f) were unacceptable inputs for the second function (g); in contrast, fg could be formed because all the outputs of the first function (g) were acceptable inputs for the second function (f).
Now we recall that the set of inputs and outputs for a function are called the domain and range, respectively. Then summarising the above discussion, we conclude that fg exists if the range (set of outputs) of the first function (g) is contained in the domain (set of inputs) of the second function f.

Rule
The composition of two functions exists if the range of the first function is contained in the domain of the second function.

In fg, g is the first function

Example 4.16
Given functions f and g, where

$$\mathrm{f}(x) = x - 5 \qquad (-\infty, \infty)$$

$$\mathrm{g}(x) = \frac{1}{x} \qquad (0, \infty)$$

determine whether fg and / or gf exist.

fg For $\mathrm{g}(x) = \frac{1}{x}$ with domain (0, ∞), the range is also (0, ∞). Now this range is contained in the domain of f (i.e. in (− ∞, ∞)) so that fg can be formed.

Then $\mathrm{fg}(x) = \mathrm{f}(\mathrm{g}(x)) = \mathrm{f}\left(\frac{1}{x}\right) = \frac{1}{x} - 5$.

We note that the domain of fg is the domain of the first function g.

gf For $\mathrm{f}(x) = x - 5$ with domain (−∞, ∞), the range is also (−∞, ∞). This range of f contains the value 0 which is unacceptable as an input for $\mathrm{g}(x) = \frac{1}{x}$. Thus gf cannot be formed.

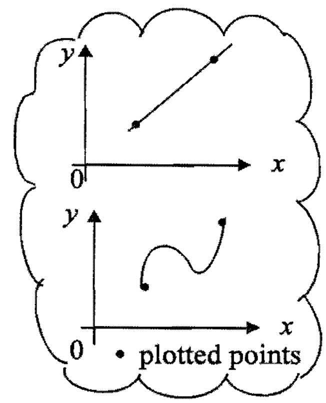
Chapter 5

Functions and Graphs : a further look

Functions whose ranges and domains are sets of real numbers can be represented by graphs. Graphs give useful insights into the behaviour of functions, enabling the identification of location of maximum and minimum values, for example.

Up until now, graphs have been produced as a result of drawing up tables of $y = f(x)$ against x and then plotting the values of x and y. Whilst this technique has proved useful it is sometimes misleading, specifically in relation to the behaviour of the graph function between the plotted points.

In this chapter, we introduce the technique of sketching graphs of functions, where we place less emphasis on plotting points and more on the features of graphs. Our approach is to consider the graphs of some basic functions and investigate how these can be used to give other graphs.



5.1 Graphs of basic functions

In this section, unless stated otherwise, all functions are defined for all values of x.

The simplest graphs are straight lines which are derived from linear equations.

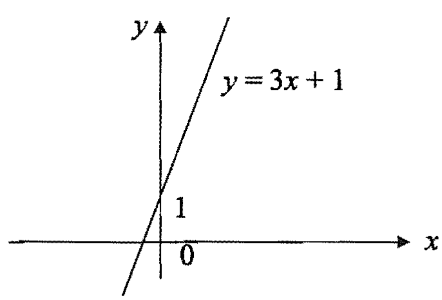
Example 5.1 Straight lines

Let's consider the function defined by

$$f(x) = 3x + 1$$

or $$y = 3x + 1.$$

Linear equations of this type were considered in **P1** where it was pointed out that such equations give rise to straight line graphs. The graph in question is shown:



Exercises 4.6

1. If $f(x) = x^2 + 2$ domain $(-\infty, \infty)$
and $g(x) = 3x - 2,$ domain $(-\infty, \infty)$
write down the rules for the functions fg and gf and give the domains and ranges in both cases.

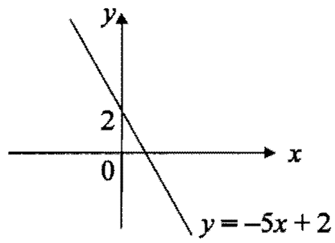
2. Given $g(x) = 3x + 1$ domain $(-\infty, \infty)$
$f(x) = 2x - 4,$ domain $(-\infty, \infty)$
find the rules for the functions fg and gf and give the domains and ranges in both cases.

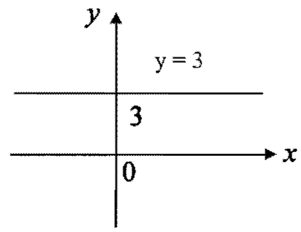
3. Given $f(x) = 2x + 1$ domain $(0, \infty)$
$g(x) = x - 1,$ domain $(-\infty, \infty)$
determine which of fg and gf exist, giving the domain and range in that case.

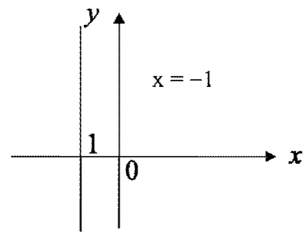
4. Functions f, g and h are defined as follows :-
$f(x) = x^2 + 3$ domain $[0, 4]$
$g(x) = \sqrt{x - 4}$ domain $(4, 20]$
$h(x) = \dfrac{2}{x^2}$ domain $(1, 15]$
State which of the following composite functions can be formed and which cannot, giving your reasons in each case.
(a) gf (b) fg (c) fh (d) hf (e) gh (f) hg

5. Given $f^{-1}(x) = -3 + \sqrt{9 + x}$ when $f(x) = x^2 + 6x$ $[-3, 3]$, show that $f^{-1}f(x) = x$.

Other examples are $y = -5x + 2$, $y = 3$, and $x = -1$.







Example 5.2 The modulus function

This function is written as

$$f(x) = +\sqrt{x^2}$$

or $$y = +\sqrt{x^2}.$$

The value of y is never negative, and is positive except when $x = 0$. The function is often written as

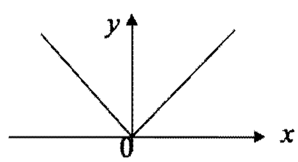
$$f(x) = |x|$$

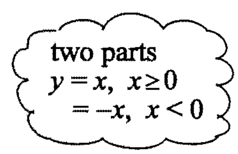
or $$y = |x|,$$

where the modulus notation | | is interpreted as the 'absolute value' (ignoring the sign) of a number. Thus

$$|3| = 3 \quad \text{and} \quad |-5| = 5.$$

The graph of $y = |x|$ is shown below.





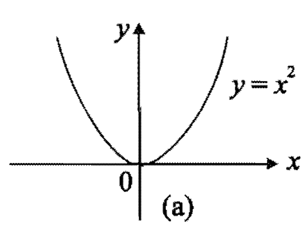
Another, possibly less familiar, way of writing the modulus function is shown in the bubble. This form is useful when using the modulus function in the branch of mathematics known as Calculus.

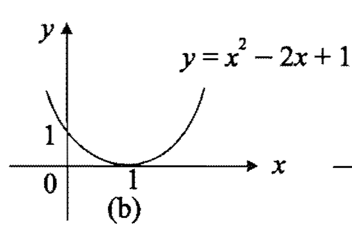
Example 5.3 Quadratic functions

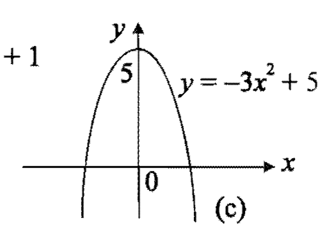
Quadratic functions of x involve terms in x^2 but no higher powers of x. Examples are $f(x) = x^2$, $g(x) = x^2 - 2x + 1$, $h(x) = -3x^2 + 5$, or in terms of y,

$$y = x^2, \quad y = x^2 - 2x + 1, \quad y = -3x^2 + 5.$$

Graphs of quadratic functions in x involve curves known as parabolas.

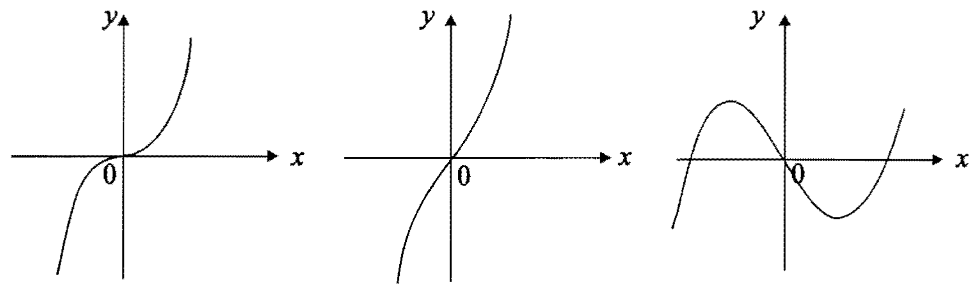






All parabolas have a common feature in that they possess a single turning point. For the cases considered here (other forms are possible) : as we move our eyes to the right, graphs (a) and (b) change from moving down the page to moving up the page, i.e. the functions change from being decreasing functions to increasing functions. In contrast, graph (c) changes from moving up the page to moving down the page, i.e. the function changes from being an increasing function to a decreasing function.
Finally, it is noted that the shapes of all the graphs are similar, (c) being an (a), (b) type graph turned upside down.

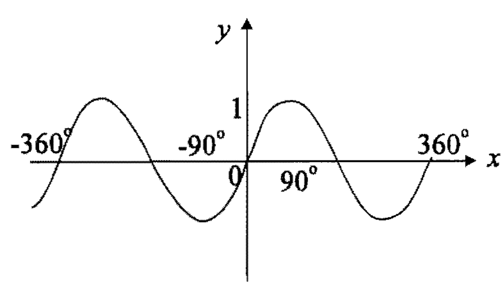
Example 5.4 Cubic functions
Here the polynomial function involves terms in x^3 but no higher powers of x. In contrast to quadratic functions where the basic shape ('cup' in (a), (b), 'cap' in (c)) is fixed, cubic functions can exhibit various shapes. Consider the graphs of (a) $y = x^3$ (b) $y = x^3 + x$ (c) $y = x^3 - x$ as shown below.



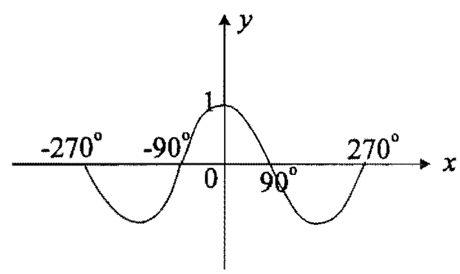
(a) $y = x^3$ (b) $y = x^3 + x$ (c) $y = x^3 - x$

All the graphs pass through (0,0) although the graph (a) is flat at that point whereas (b) and (c) are not.
In (c) there are two turning points of the type occurring with quadratic functions.

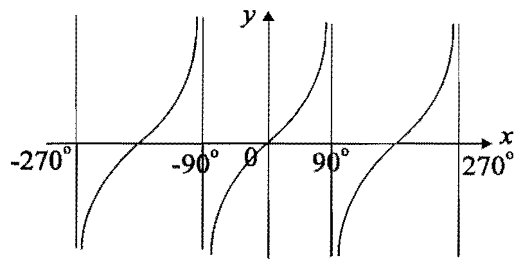
Example 5.5 Trigonometric functions
These type of functions were introduced in **P1**. For completeness the graphs of sin x, cos x and tan x are displayed here.



$y = \sin x$



$y = \cos x$



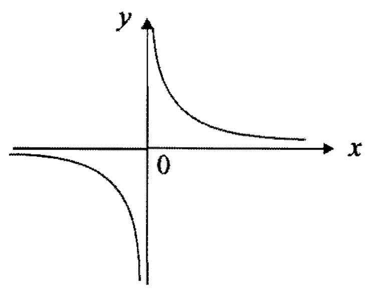
$y = \tan x$

Example 5.6 The reciprocal function

We consider $\quad f(x) = \dfrac{1}{x}$

or $\quad y = \dfrac{1}{x}.$

The function is undefined for $x = 0$. As x approaches 0 through positive values of x e.g. ($x = 1, 0.1, 0.01, 0.001$ etc) y takes increasingly large positive values (1, 10, 100,1000 etc), and as x approaches 0 through negative values (-1, -0.1, -0.01,-0.001) y takes increasingly large negative values (-1, -10, -100, -1000 etc). As x takes large values (positive or negative) y takes small (positive or negative) values. The graph is as shown below.



The function is seen to be discontinuous at $x = 0$. In passing, it should be noted for $f(x) = \dfrac{1}{x}$, $f\left(-\dfrac{1}{2}\right) = -2$, and $f\left(\dfrac{1}{2}\right) = 2$ but it cannot be deduced that $f(x) = 0$ for some x in $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$.

Example 5.7 The exponential function

We consider functions such as

$$f(x) = 2^x,\ g(x) = \left(\frac{1}{3}\right)^x,\ h(x) = 4^{5x-1}$$

$$\text{or } y = 2^x, \quad y = \left(\frac{1}{3}\right)^x, \quad y = 4^{5x-1},$$

where x occurs in the exponent. Such functions occur frequently in mathematics and must therefore be included in our catalogue of basic functions.

Here, we draw up a table of values (in spite of the reservations expressed earlier), and plot the graphs of $y = 2^x$ and $y = \left(\frac{1}{3}\right)^x$.

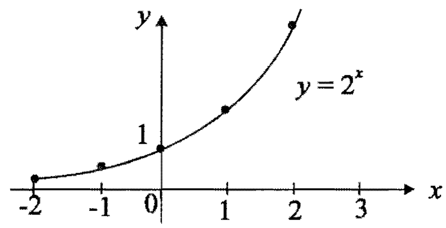
(a)

x	-2	-1	0	1	2	3
$y = 2^x$	$\frac{1}{2^2} = \frac{1}{4}$	$\frac{1}{2}$	$2^0 = 1$	2	4	8

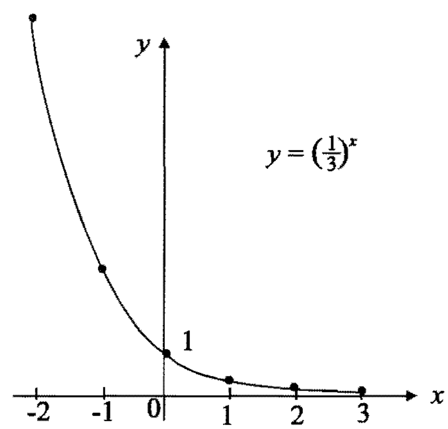
(b)

x	-2	-1	0	1	2	3
$y = \left(\frac{1}{3}\right)^x$	$\left(\frac{1}{3}\right)^{-2} = 9$	$\left(\frac{1}{3}\right)^{-1} = 3$	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$

(a)



(b)



It is assumed that the curves are accurately represented by joining the calculated points.

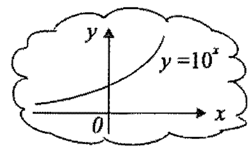
One common feature of the graphs is that $y > 0$ for all points on both graphs, i.e. the graphs are above the x-axis. The graphs differ in that (a) is an increasing function (the graph climbs to the right) and (b) is a decreasing function (the graph falls to the right). Other features of the graphs are listed below.

	(a)	(b)
(i)	$y = 1$ when $x = 0$	$y = 1$ when $x = 0$
(ii)	$y \to \infty$ as $x \to \infty$	$y \to 0$ as $x \to \infty$
(iii)	$y \to 0$ as $x \to -\infty$	$y \to \infty$ as $x \to -\infty$
(iv)	the graph becomes steeper to the right	the graph becomes steeper to the left.

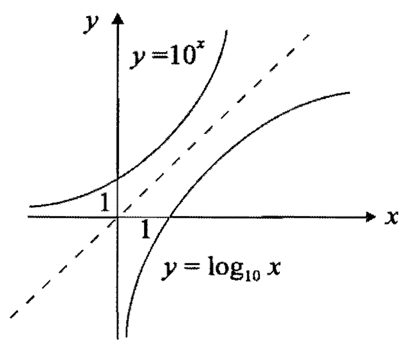
A little thought shows that the contrasting features occur for $y = a^x$ according to whether a is greater than or less than 1.

$y = a^x$, is only defined for $a > 0$.

Example 5.8 The log function
Let's return briefly to the exponential function. To fix ideas, we consider $f(x) = 10^x$.
The function is one-one and therefore its inverse function f^{-1} exists and in fact is defined by $f^{-1}(x) = \log_{10} x$.



The graph of $\log_{10} x$ is easily plotted by first using the $\log_{10}$ button to calculate values, or by reflecting the graph of $y = 10^x$ in the line $y = x$.



From the graph of $y = \log_{10} x$ the following features are apparent :-

(a) $\log_{10} 1 = 0$,

(b) $\log_{10} x \to -\infty$ as $x \to 0$,

(c) $\log_{10} x \to \infty$ as $x \to \infty$

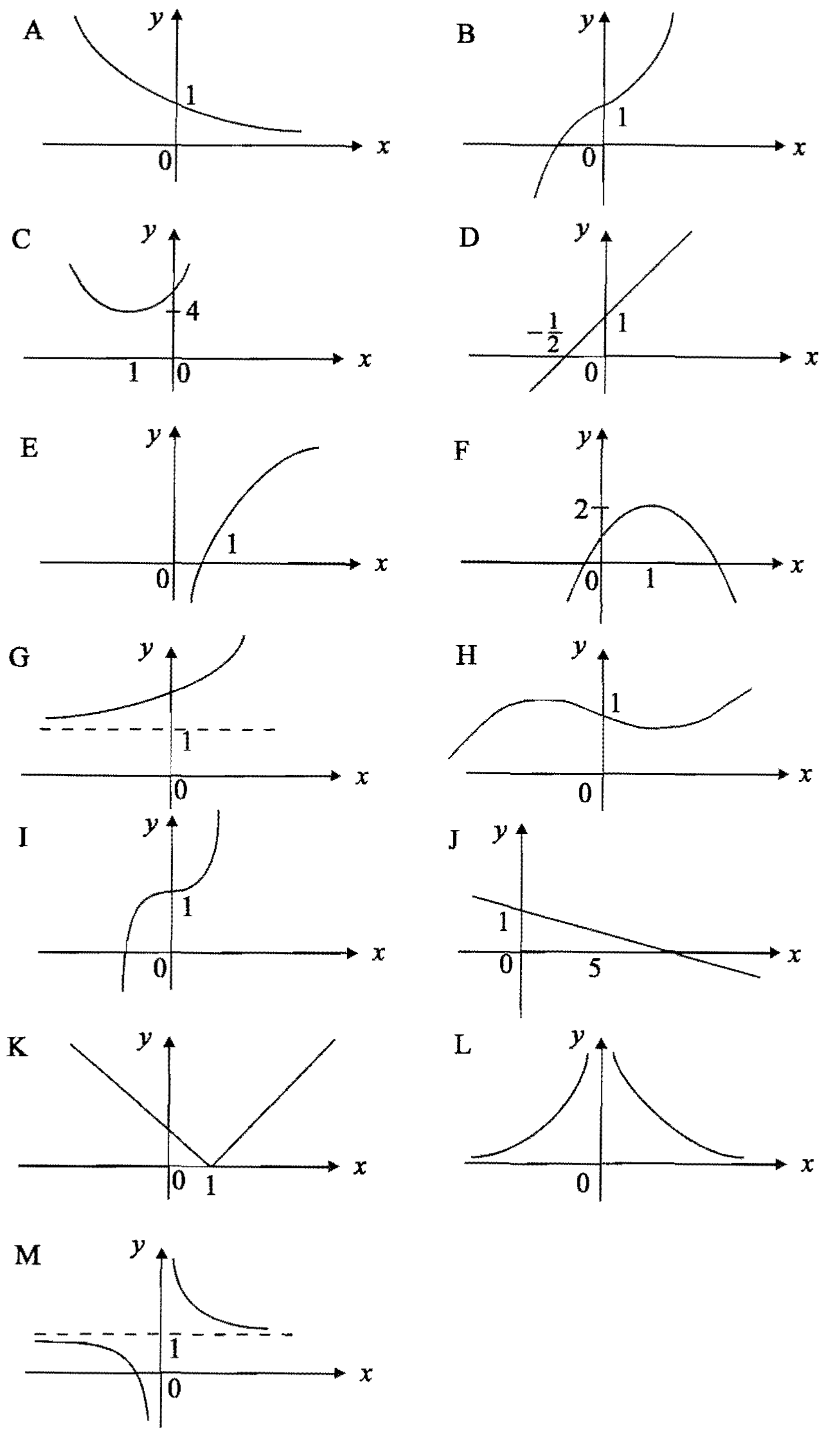
and $\log_{10} x$ has domain $(0, \infty)$.

Exercises 5.1

1 The following graphs relate to the following :-

(i) $y = x^2 + 2x + 5$	(ii) $y = 2x + 1$
(iii) $y = x^3 + 1$	(iv) $y = -x^2 + 2x + 1$
(v) $y = x^3 - x + 1$	(vi) $y = (1.3)^x + 1$
(vii) $y = x^3 + x + 1$	(viii) $y = (0.2)^x$
(ix) $y = \log_{10} x$	(x) $y = \dfrac{1}{x} + 1$
(xi) $y = \lvert x - 1 \rvert$	(xii) $y = \dfrac{1}{\lvert x \rvert}$
(xiii) $y = -\dfrac{1}{5}x + 1$.	

By first considering the shape of some of the graphs in **Examples 5.1-5.8**, group the equations and graphs as $((i), B)$ for example (this is not necessarily the correct result!).



2 Sketch the graphs of $y = 3 - x$ and $y = \log_{10} x$. Hence show that there is only one value of x satisfying

$$x + \log_{10} x - 3 = 0$$

3 Show by sketching appropriate graphs that the equation $e^x + x - 2 = 0$ has only one root and that is positive.

4 Sketch the graphs of $y = \sin x$ and $y = x - 2$ and show that $\sin x - x + 2 = 0$ has only one root.

5.2 Effects of transformations on the graph of $y = f(x)$

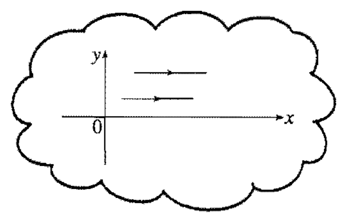
In this section we consider the effects of a number of processes on the graph of $y = f(x)$. These processes and others are known collectively as transformations of the *xy* plane. Specifically, we consider the transformations known as translations and scalings.

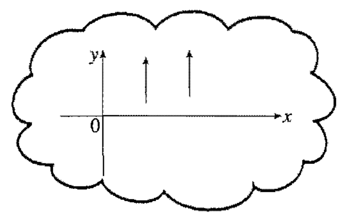
1. **Translation**

This relates to situations where every point moves by the same amount. There are two cases considered, namely

(i) a movement or translation of all points in the *xy* plane, through the same distance parallel to the *x* axis,

(ii) a movement or translation of all points in the *xy* plane parallel to the y direction.

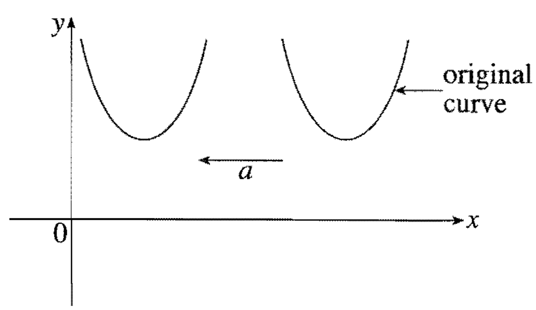




(i) Translation in *x* direction

For a reason which will become apparent, it is convenient to suppose that all points move a distance $-a$ in the *x* direction, where *a* may be positive or negative.

Geometrically, the effect of the translation $-a$ in the *x* direction is to move the curve $y = f(x)$ bodily in the *x* direction, as shown. In the diagram *a* is taken to be positive.



Given that the original curve has equation $y = f(x)$, what is the equation of the new curve?

Now the effect of the translation

$(x, y) \longrightarrow (x-a, y)$ is to form new co-ordinates X, Y given by

$$X = x - a,$$
$$Y = y.$$

X decreased by a

Then $x = X + a$ and $y = Y$.
Substitution for x, y in $y = f(x)$ gives

$$Y = f(X + a).$$

Now let's drop the capital letters because this relation is equally valid with other letters. Then we obtain.

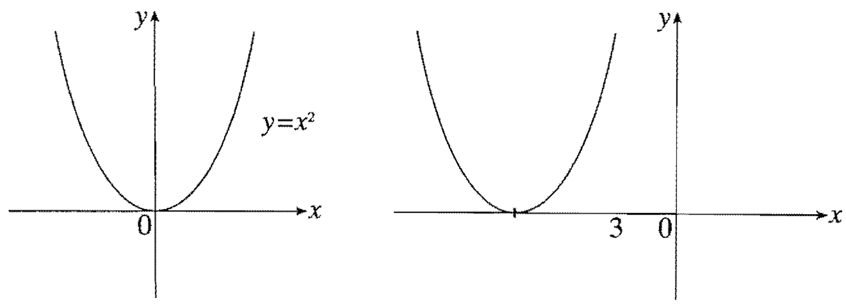
$$y = f(x + a).$$

Note the possibly surprising result:
a translation of $-a$ in the x direction changes x to $x + a$ in $f(x)$. That was the reason for our choice of $-a$ in the x-translation.

Rule I

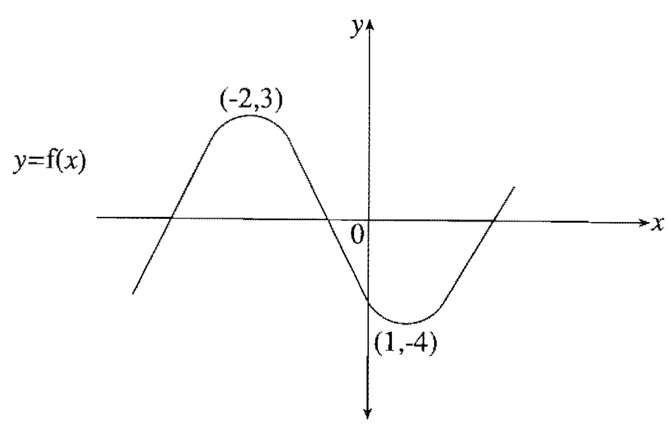
A translation of $-a$ in the x direction converts the curve $y = f(x)$ into $y = f(x + a)$, referred to the original axes.

Example 5.9
Draw the graphs of $y = x^2$ and $y = (x + 3)^2$. The graphs are as shown.

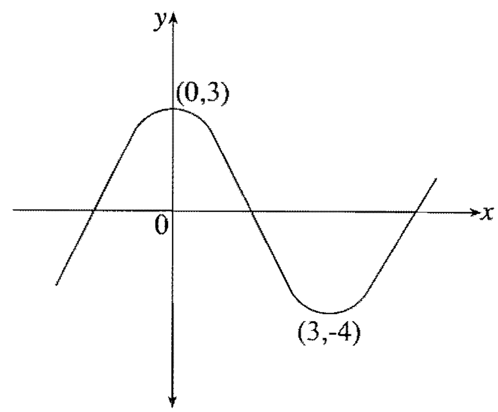


Now $f(x) = x^2$, $f(x + 3) = (x + 3)^2$ so that $a = 3$. The second graph is obtained from the first by a translation of -3 along the x-axis.

Example 5.10
Given $y = f(x)$ has the graph shown, sketch the graph of $y = f(x - 2)$.

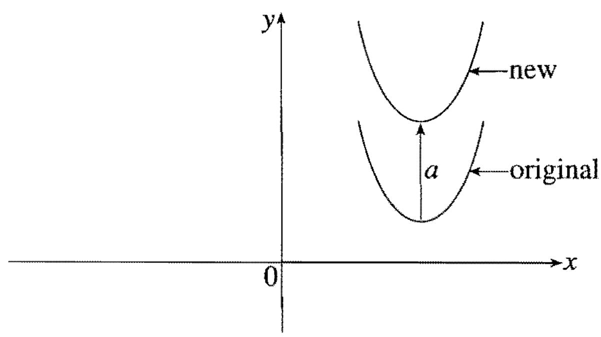


Here $a = -2$ and the new graph is obtained by moving the original graph through $-(-2) = 2$ along the x direction. The stationary points $(-2, 3)$, $(1, -4)$ are translated to $(0, 3)$ and $(3, -4)$ respectively. The transformed graph is therefore as shown.



(ii) <u>Translation in y direction</u>

Geometrically the effect of the translation a in the y direction is to move the curve $y = f(x)$ a distance a in the y direction (note we do not use $-a$ here).



Then the transformation $(x, y) \longrightarrow (x, y + a)$ defines new coordinates X, Y given by

$$X = x,$$
$$Y = y + a.$$

Then $x = X, \;\; y = Y - a.$

Substitution for x and y in $y = f(x)$ gives

$$Y - a = f(X)$$

or $$Y = f(X) + a.$$

Dropping the capitals, we have

$$y = f(x) + a.$$

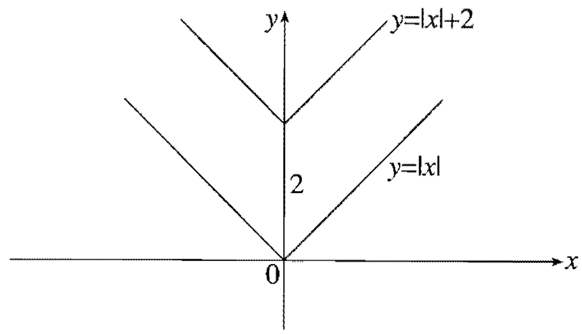
Rule II

A translation of a in the y direction converts the curve $y = f(x)$ into $y = f(x) + a$, referred to the original axes.

Example 5.11

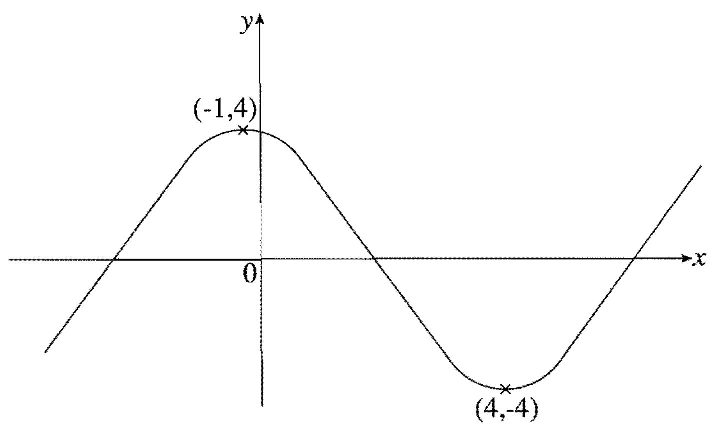
Draw the graphs of $y = |x|$ and $y = |x| + 2$.

Here $f(x) = |x|$ and $a = 2$. The graphs are as shown, where the original graph of $y = |x|$ has been moved a distance of 2 in the y direction.

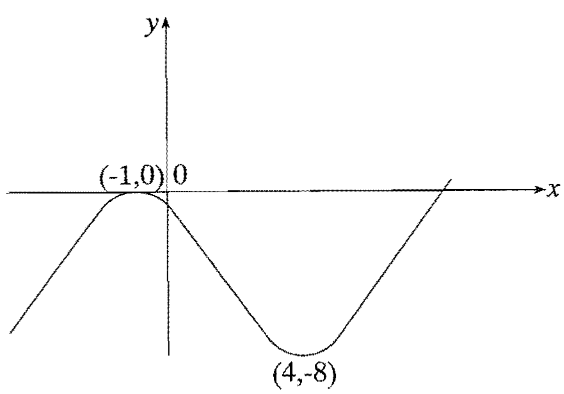


Example 5.12

Given that the graph of $y = f(x)$ is as shown, find the graph of $y = f(x) - 4$.



The new graph is found by moving the graph of $y = f(x)$ through -4 along the y direction. The original stationary points change to $(-1, 0)$ and $(4, -8)$.



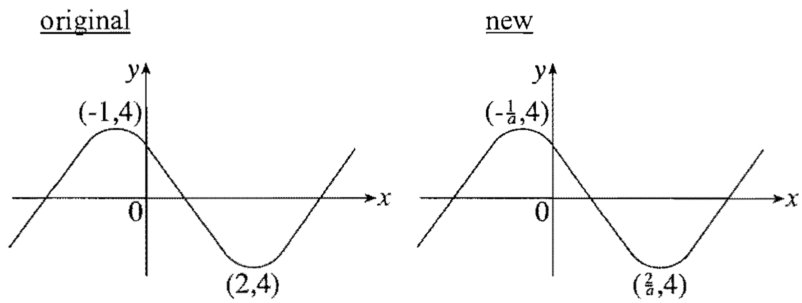
2. **Scaling**

This relates to situations where distances are multiplied by a constant factor. As with translations, we consider scalings in the x and y directions separately.

(i) Scaling in the x direction

For a reason which will become apparent, it is convenient to suppose that there is a scaling of $\frac{1}{a}$ in the x direction, i.e. all distances are multiplied by $\frac{1}{a}$.

The geometrical effect on a graph is shown below.



In particular, the x–coordinates of the stationary points of the original curve have been multiplied by $\frac{1}{a}$. To find the equation of the new graph we proceed as previously.

The transformation $(x, y) \longrightarrow \left(\frac{1}{a}x, y\right)$

defines new coordinates X, Y given by

$$X = \frac{1}{a}x, \quad Y = y.$$

Then $x = aX, \quad y = Y$

and $y = f(x)$ becomes after substitution for x and y:-

$$Y = f(aX).$$

On dropping the capitals, we have

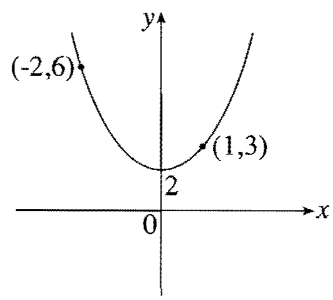
$$y = f(ax).$$

Note again a possibly surprising result: a scaling of $\frac{1}{a}$ in the x direction changes x to ax in $f(x)$.

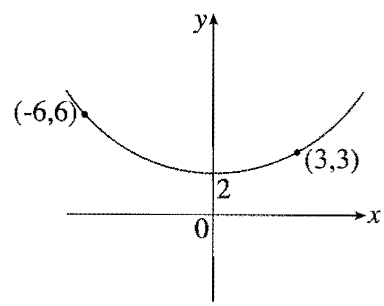
Rule III

A scaling of $\frac{1}{a}$ in the x direction converts the curve $y = f(x)$ into $y = f(ax)$.

Example 5.13
The graph of y = $x^2 + 2$ is as shown.



Then a scaling of 3 in the x direction gives the following graph, the effect on the points (–2, 6) and (1, 3) of the original graph beiñg indicated.



Noting that the scaling $\frac{1}{a} = 3$, we have $a = \frac{1}{3}$.

Now $f(x) = x^2 + 2$ so that

$$f(ax) = f\left(\frac{1}{3}x\right) = \left(\frac{1}{3}x\right)^2 + 2 = \frac{1}{9}x^2 + 2 .$$

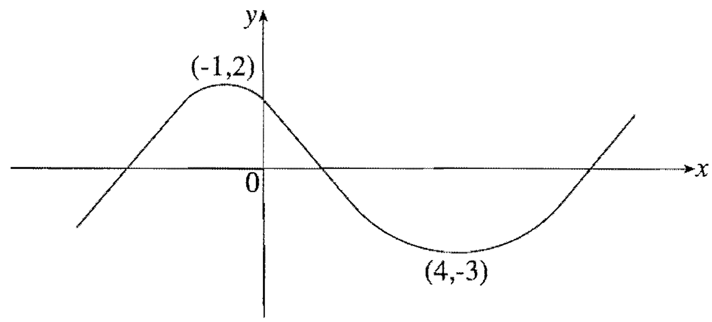
Thus the equation of the transformed graph

is $\qquad y = \frac{1}{9}x^2 + 2$

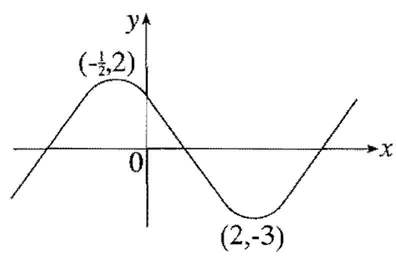
or $\qquad 9y = x^2 + 18 .$

Example 5.14

Given that $y = f(x)$ has the graph shown, sketch the graph of $y = f(2x)$.



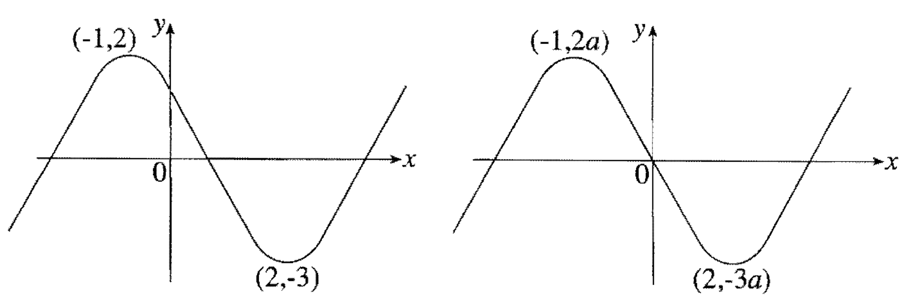
Here $a = 2$ and the transformation is an x scaling of $\frac{1}{2}$. The transformed graph is as shown, the original stationary points being transformed into $(-\frac{1}{2}, 2)$ and $(2, -3)$



ii) Scaling in the y direction

It is supposed that there is a scaling of a in the y direction, i.e. all distances in that direction are multiplied by a.

The geometrical effect on a graph is shown below.



In particular, the y coordinates of the stationary points have been multiplied by a. To find the equation of the new graph we proceed as before.

The transformation $(x, y) \longrightarrow (x, ay)$
defines new coordinates (X,Y) given by

$$X = x, \quad Y = ay.$$

Then $x = X,\ y = \dfrac{Y}{a}$ so that $y = f(x)$

becomes $$\frac{Y}{a} = f(X)$$

or $$Y = a\,f(X).$$

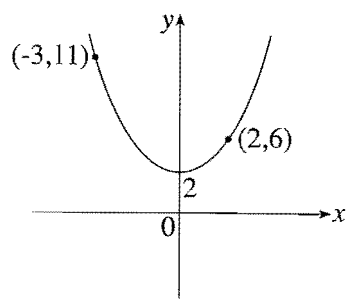
Changing the capital letters, we have

$$y = a\,f(x).$$

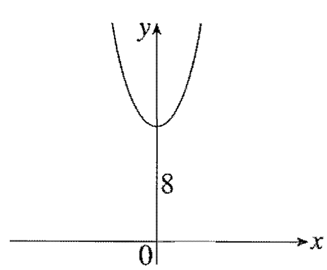
Rule IV

A scaling of a in the y direction converts the curve $y = f(x)$ into $y = a\,f(x)$.

Example 5.15
The graph of $y = x^2 + 2$ is as shown.



A scaling of 4 in the y direction gives the following graph, the effect on the intercept on the x-axis being indicated.

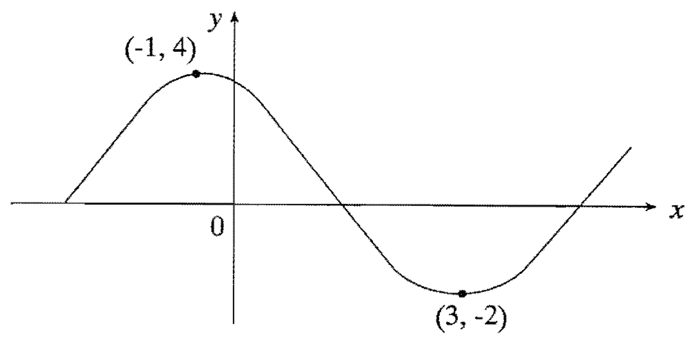


Noting that the scaling in the y direction is $a = 4$ and $f(x) = x^2 + 2$, we see that the equation is

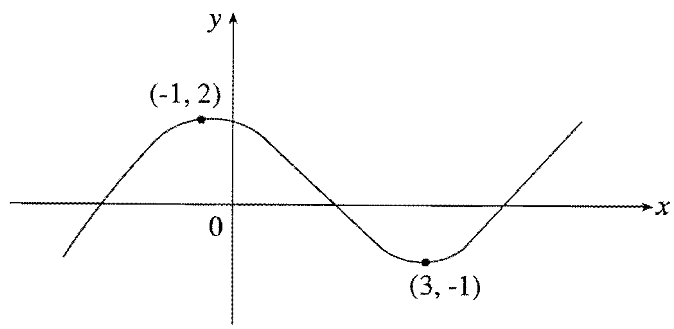
$$y = 4\,f(x) = 4(x^2 + 2)$$

or $$y = 4x^2 + 8.$$

Example 5.16
Given that $y = f(x)$ has the graph shown, find the graph arising as a result of a scaling of $\frac{1}{2}$ in the y direction being applied.



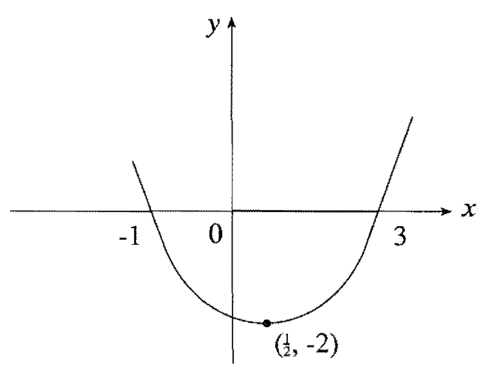
The transformed graph is as shown.



The stationary points $(-1, 4)$ and $(3, -2)$ have been transformed into $(-1, 2)$ and $(3, -1)$ respectively.

Exercises 5.2

1.



The sketch shows the graph of $y = f(x)$. The curve passes through $(-1, 0)$ and $(3, 0)$, and has a minimum point at $\left(\frac{1}{2}, -2\right)$.

Sketch, on separate diagrams, the graphs of
(a) $y = f(x+2)$ (b) $y = f(x) + 2$ (c) $y = f(3x)$

2. Sketch the graph of $y = \frac{1}{x}$ and the graph resulting from the translation $(x, y) \longrightarrow (x-1, y)$ followed by the scaling $(x, y) \longrightarrow (x, 2y)$. What is the equation of the graph resulting from these transformations?

3. Sketch the graph of $y = |x|$. What x and y translations transform $y = |x|$ into $y = |x-2| - 4$? Sketch the second graph.

4 Sketch the graphs of $y = \cos x$ and $y = \sin x$. Show that $y = \cos x$ and $y = \sin\left(x + \frac{\pi}{2}\right)$ have the same graph.

5 Use the graph of $y = \sin x$ to sketch the graph of $y = 5\sin 3x + 4$.

6 Sketch the graph of $y = 3^x$.
Use this graph to sketch the graph of $y = 2 \times 3^x + 5$.

7 Given that $y = \left(\frac{1}{2}\right)^{2x+1} + 4$ can be written as $y = \frac{1}{2}\left(\frac{1}{2}\right)^{2x} + 4$, sketch its graph starting from the graph of $y = \left(\frac{1}{2}\right)^x$.

8 Sketch the graphs of
(i) $y = \log_{10} x$ (ii) $y = 3\log_{10} x$ (iii) $y = 3\log_{10}(x) + 5$
(iv) $y = 3\log_{10}(2x) + 5$.

Chapter 6

Cartesian Coordinate Geometry of the Circle

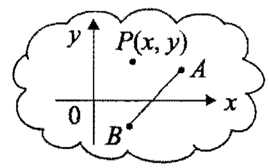
In **P1**, the properties of straight lines were investigated by algebraic methods. Here we make a start in applying algebra to the study of curves.

6.1 Locus of a point

The locus of a point is the path of the point when it moves under certain conditions. The locus or path may often be described by equations in coordinate geometry.

Example 6.1

A point $P(x, y)$ moves such that it is always equidistant from the points $A(5, 1)$ and $B(3, -1)$. Find the equation of the locus of P.



Expressed geometrically, the condition satisfied by the point P is $\quad AP = BP$.

Now $\quad AP^2 = (x-5)^2 + (y-1)^2,$

$\quad BP^2 = (x-3)^2 + (y+1)^2.$

Then $AP = BP$ is equivalent to

$$AP^2 = BP^2$$

which leads to

$$(x-5)^2 + (y-1)^2 = (x-3)^2 + (y+1)^2.$$

$$\therefore\ x^2 - 10x + 25 + y^2 - 2y + 1 = x^2 - 6x + 9 + y^2 + 2y + 1$$

so that $\quad 4x + 4y - 16 = 0$

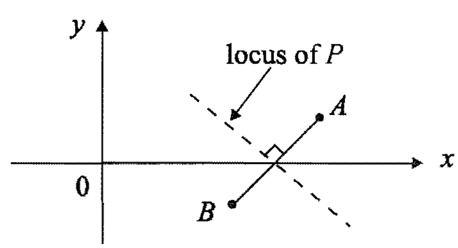
or $\quad x + y - 4 = 0.$

We recognise this as the equation of a straight line.

In fact, the equation describes the line passing through the mid point of AB which is perpendicular to AB.

$(x_1 - x_2)^2 + (y_1 - y_2)^2$, **P1.**

$\alpha x + \beta y + \gamma = 0$, **P1.**



Because of the close connection between the locus and the equation satisfied by points lying on the locus, we refer to the equation itself as the locus.

Example 6.2
Find the locus of a point P whose distance from the point $A(1, -2)$ is twice the distance from the origin O.

Let $P(x, y)$ be the point on the locus.

Then since $PA = 2PO$,

$PA^2 = 4PO^2$

so that $(x-1)^2 + (y+2)^2 = 4(x^2 + y^2)$.

$\therefore$ $x^2 - 2x + 1 + y^2 + 4y + 4 = 4x^2 + 4y^2$

giving $3x^2 + 3y^2 + 2x - 4y - 5 = 0$.

You are not expected to recognise this type of equation at this stage.

Example 6.3
Find the locus of a point P such that PA is perpendicular to PB where A is $(0,1)$ and B is $(0, -1)$.

Let $P(x, y)$ be a point on the locus.

The gradient of PA is $\dfrac{y-1}{x-0} = \dfrac{y-1}{x}$.

The gradient of PB is $\dfrac{y+1}{x-0} = \dfrac{y+1}{x}$.

Since the lines are perpendicular, the product of their gradients is -1.

Then $\dfrac{y-1}{x} \times \dfrac{y+1}{x} = -1$.

$\therefore$ $y^2 - 1 = -x^2$

or $x^2 + y^2 = 1$.

Again, you are not expected to recognise this curve.

Example 6.4
Find the locus of a point which moves so that its distance from the point $A(1, 2)$ is 2.

Let $P(x, y)$ be a point on the locus.

Then $PA = 2$

so that $PA^2 = 4$.

$\therefore$ $(x-1)^2 + (y-2)^2 = 4$

so that $x^2 - 2x + 1 + y^2 - 4y + 4 = 4$.

$\therefore$ $x^2 + y^2 - 2x - 4y + 1 = 0$.

Exercises 6.1

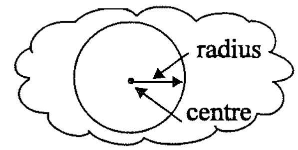
1. Find the locus of a point which moves so that its distance from the point $A(2, 0)$ is three times its distance from the origin O.
2. A point $P(x, y)$ moves so that its distance from the origin is 5. Find the equation of the locus of the point.
3. Write down the distance of the point (x, y) from the line $y = -1$. Find the locus of a point which is equidistant from the origin O and the line $y = -1$.
4. A is the point $(-1, 2)$ and B is the point $(1, -2)$. A point P moves so that AP and PB are perpendicular. Find the locus of P.

5. Find the locus of a point which moves so that it is equidistant from the point $(a, 0)$ and the line $x = -a$.

6. Find the locus of a point which moves so that its distance from the point $(a, 0)$ is three times its distance from the line $x = -a$.

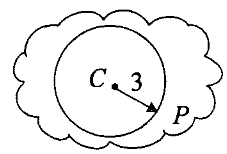
6.2 The Circle

A circle is the locus of a point which moves in a plane so that its distance from a fixed point in the plane is constant. The fixed point is called the <u>centre</u> and the constant distance the <u>radius</u>.



Example 6.5

Find the equation of the circle having centre $C(1, -2)$ and radius 3.

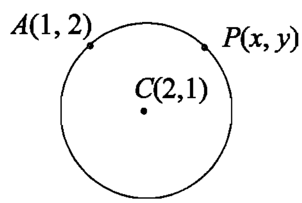


Let $P(x, y)$ be a point on the circle.

$$\begin{aligned} \text{Then} \quad CP &= 3 \\ \text{gives} \quad CP^2 &= 9 \\ \text{so that} \quad (x-1)^2 + (y+2)^2 &= 9. \\ \therefore \quad x^2 - 2x + 1 + y^2 + 4y + 4 &= 9 \\ \text{giving} \quad x^2 + y^2 - 2x + 4y - 4 &= 0. \end{aligned}$$

Example 6.6

Find the equation of the circle having centre $C(2,1)$ which passes through the point $A(1, 2)$.

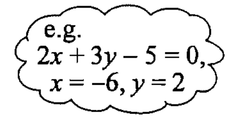


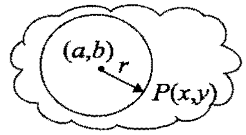
Whilst the radius of the circle is not given, it can be calculated by using the points A and C. Thus, when $P(x, y)$ is on the circle,

$$\begin{aligned} CP^2 &= CA^2 \\ \text{so that} \quad (x-2)^2 + (y-1)^2 &= (1-2)^2 + (2-1)^2. \\ \therefore \quad (x-2)^2 + (y-1)^2 &= 2 \\ \text{or} \quad x^2 + y^2 - 4x - 2y + 3 &= 0. \end{aligned}$$

<u>**The standard equation of a circle**</u>

In **P1**, it was pointed out that to represent a straight line an equation must be of first degree in x and y (when x and y appear, at least). Is it possible to make a similar statement in relation to the equation of a circle? To answer this question, let's derive the standard equation of a circle.





The most general equation of a circle is

$$(x-a)^2+(y-b)^2 = r^2$$

which reduces to

$$x^2+y^2-2ax-2by+a^2+b^2-r^2 = 0.$$

This suggests that every equation of the form

$$x^2+y^2+2gx+2fy+c = 0$$

represents a circle.

This equation may be put in the form

$$(x+g)^2+(y+f)^2 = g^2+f^2-c$$

so that $(-g, -f)$ is the centre and $\sqrt{g^2+f^2-c}$ is the radius of the circle.

In summary, the general equation of a circle is

The coefficients of x^2 and y^2 are equal and there is no term in xy.

Check this by multiplying out factors.

$x^2+y^2+2gx+2fy+c = 0$ with centre $(-g, -f)$ and radius $\sqrt{g^2+f^2-c}$.

Example 6.7

Determine which of the following equations represent circles. Where the equation describes a circle state its centre and radius.

(a) $3x+y-5 = 0$ (b) $y^2 = 4x$ (c) $x^2+y^2+4x-2y+1=0$
(d) $x^2+y^2+3xy-4y+3=0$ (e) $3x^2+3y^2=5$
(f) $3y^2+x^2-y=2$ (g) $2x^2+2y^2-6x-5y=0$
(h) $3x^2+3y^2+6x-5=0$ (i) $2x^2-2y^2=5$

(a) A straight line.
(b) Curve but not a circle.
(c) Circle with centre $(-2, 1)$, radius 2.
(d) Curve but not a circle (presence of xy).
(e) Circle with centre (0, 0), radius $\sqrt{\frac{5}{3}}$.
(f) Curve but not a circle (coefficients of x^2 and y^2 are unequal)
(g) Circle with centre $\left(\frac{3}{2},\frac{5}{4}\right)$, radius $\frac{\sqrt{61}}{4}$.
(h) Circle with centre $(-1, 0)$, radius $\sqrt{\frac{8}{3}}$.
(i) Curve but not a circle (coefficients of x^2 and y^2 are unequal).

Exercises 6.2

1 Find the equations of the circles with the following centres and radii (plural of radius).

(a) (0, 1); 3 (b) $(-1, 2)$; $\sqrt{5}$ (c) (2, 3); 4
(d) $(-1, -1)$; $\sqrt{2}$ (e) (4, 1); $\sqrt{5}$

2. Find the centres and radii of the following circles :-

(a) $x^2+y^2+4x+2y+4 = 0$ (b) $x^2+y^2-2x-4y-4 = 0$
(c) $x^2+y^2-3y = 12$ (d) $x^2+y^2-4x = 0$

(e) $4x^2 + 4y^2 - 8x - 7y = 2$ (f) $4x^2 + 4y^2 = 9$

3. Find the equation of the circle with centre (2, − 1) which passes through the point (2, 1).

4. Find the equation f the circle which passes through the points (0, 4), (0, 9) and (6, 0). (Let the equation be $x^2 + y^2 + 2gx + 2fy + c = 0$).

5. Find the equation of the circle which has the line joining $A(1, 2)$ and $B(-1, 3)$ as a diameter.
(The mid-point of a diameter is the centre of the circle).

6. A circle of centre O has equation

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

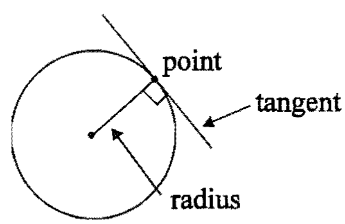
The point $P(\alpha, \beta)$ lies outside the circle.

(a) Write down the coordinates of O and the radius of the circle.
(b) Find an expression for OP.
(c) A tangent to the circle from P intersects the circle at T. Show that

$$PT^2 = \alpha^2 + \beta^2 + 2g\alpha + 2f\beta + c.$$

The equation of a tangent to a circle

The equation of a tangent to any curve at a point can be found by calculus. Here, we shall not use calculus but instead exploit the particular geometry of the circle. In particular, let's recall that a tangent to a circle is perpendicular to the radius at the point.



Example 6.8

Verify that the point (3, 5) lies on the circle

$$x^2 + y^2 - 4x - 6y + 8 = 0$$

and find the equation of the tangent at this point.

If the point lies on the circle, its coordinates must satisfy the equation. Substitution of $x = 3$, $y = 5$ in the equation gives

$$\begin{aligned}\text{left hand side} &= 3^2 + 5^2 - 4(3) - 6(5) + 8\\ &= 9 + 25 - 12 - 30 + 8\\ &= 0 = \text{right hand side.}\end{aligned}$$

∴ The point (3, 5) lies on the circle.

The centre of the circle is (2, 3) so that the gradient of the radius to the point (3, 5) is

$$\frac{5-3}{3-2} = 2,$$

and therefore the gradient of the tangent is $-\dfrac{1}{2}$.

Thus the equation of the tangent is

$$y - 5 = -\frac{1}{2}(x - 3).$$

$$\therefore \qquad 2y - 10 = -x + 3$$

so that $\quad 2y + x - 13 = 0.$

Product of the gradients of perpendicular lines is – 1, P1.

Example 6.9

(a) Find the equation of the circle passing through the origin O and the points $A(1, 0)$ and $B(0, 1)$.

(b) Find the equation of the tangents to the circle at B and $P(1, 1)$.

(c) The tangents at B and P meet at Q. Prove that the length PQ is equal to the radius of the circle.

(a) Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

Since the circle passes through (0, 0), we have

$$0^2 + 0^2 + 2g(0) + 2f(0) + c = 0.$$

$$\therefore \qquad c = 0.$$

Similarly, since $A(1, 0)$ and $B(0, 1)$ lie on the circle :-

$$1^2 + 0^2 + 2g(1) + 2f(0) = 0,$$

$$0^2 + 1^2 + 2g(0) + 2f(1) = 0,$$

which reduce to

$$1 + 2g = 0$$

$$1 + 2f = 0.$$

$$\therefore \qquad g = -\frac{1}{2},\; f = -\frac{1}{2}.$$

The equation of the circle is therefore

$$x^2 + y^2 + 2\left(-\frac{1}{2}\right)x + 2\left(-\frac{1}{2}\right)y + c = 0$$

or

$$x^2 + y^2 - x - y = 0.$$

(b) The gradient of the radius at $B(0,1)$ is

$$\frac{1 - \frac{1}{2}}{0 - \frac{1}{2}} = -1.$$

Centre is at $(-g, -f)$ or $\left(\frac{1}{2}, \frac{1}{2}\right)$

The gradient of the tangent at B is therefore given by

$$\text{gradient} \times -1 = -1$$

so that $\quad \text{gradient} = 1.$

The equation of the tangent at B is therefore given by

$$y - 1 = 1(x - 0)$$

or $\quad y - x - 1 = 0. \qquad (1)$

Similarly, the gradient of the radius at $P(1, 1)$ is

$$\frac{1 - \frac{1}{2}}{1 - \frac{1}{2}} = 1.$$

The gradient of the tangent at P is therefore -1.

The equation of the tangent at $P(1, 1)$ is then

$$y - 1 = -1(x - 1)$$

or $$y + x - 2 = 0. \qquad (2)$$

(c) The coordinates of Q, the point of intersection of the tangents, satisfy equations (1) and (2).

$$y - x - 1 = 0, \qquad (1)$$
$$y + x - 2 = 0. \qquad (2)$$

Addition of (1) and (2) gives

$$2y - 3 = 0$$

$\therefore$ $$y = \frac{3}{2}.$$

Substitution for y in (1) gives

$$\frac{3}{2} - x - 1 = 0$$

giving $$x = \frac{1}{2}.$$

Checking in (2), $\frac{3}{2} + \frac{1}{2} - 2 = 0$

$\therefore$ Q is $\left(\frac{1}{2}, \frac{3}{2}\right)$.

Then $PQ^2 =$

$$\left(1 - \frac{1}{2}\right)^2 + \left(1 - \frac{3}{2}\right)^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

so $$PQ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

$\sqrt{g^2 + f^2 - c}$ with $g = -\frac{1}{2}$, $f = -\frac{1}{2}$, $c = 0$.

From the equation of the circle,

$$x^2 + y^2 - x - y = 0$$

the centre is $\left(\frac{1}{2}, \frac{1}{2}\right)$ and the radius is

$$\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 - 0} = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}.$$

$\therefore$ The length PQ is equal to the radius of the circle.

Exercises 6.3

1. Verify that the given points lie on the circles and find the equations of the tangents at the points.
   (a) $(2, 2)$; $x^2 + y^2 = 8$
   (b) $(1, 1)$; $x^2 + y^2 + 4x + 2y = 8$
   (c) $(3, -1)$; $x^2 + y^2 + 2x + 4y - 12 = 0$
   (d) $(1, -1)$; $2x^2 + 2y^2 + 5x + 8y - 1 = 0$.

2. The tangent to the circle $x^2 + y^2 - 4x - 2y - 8 = 0$ at the point $(-1, 3)$ meets the x-axis at A. Find the distance of A from the centre of the circle.

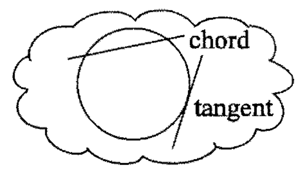
3. Find the equations of the tangents to the circle $x^2 + y^2 - 4x + 6y + 5 = 0$ at the points where it meets the y-axis.

4. The tangent to the circle $x^2 + y^2 - 2x + 4y - 15 = 0$ at the point $(-1, 2)$ meets the x and y axes at A and B, respectively. Find the coordinates of A and B. Deduce the area of triangle AOB, where O is the origin.

5. (a) Find the equations of the tangents to the circle $x^2 + y^2 - 8y + 8 = 0$ at the points $A(-2, 2)$ and $B(2, 2)$.
(b) Show that these tangents to the circle intersect at the origin O. Show that $ACBO$ is a square, where C is the centre of the circle.

The condition for a line to be a tangent to the circle

When a line intersects a circle there are two possibilities :-
(a) the line meets the circle in two points, forming a chord of the circle,
(b) the line is a tangent to the circle, touching the circle or meeting it at two coincident points.



The points of intersection are found by solving simultaneous equations.

Example 6.10

Find the points of intersection A and B of the line

$$y - x + 2 = 0$$

and the circle

$$x^2 + y^2 - 2x + 2y - 6 = 0$$

and find the length of the chord AB.

Let's solve the simultaneous equations.
From the equation of the straight line,

$$y = x - 2.$$

Substitution into the equation of the circle gives

$$x^2 + (x-2)^2 - 2x + 2(x-2) - 6 = 0$$

which reduces to

$$2x^2 - 4x - 6 = 0.$$

$$\therefore \quad x^2 - 2x - 3 = 0.$$

$$\therefore \quad (x - 3)(x + 1) = 0.$$

Thus $\quad x = 3, -1.$

Substitution of these values of x into

$$y = x - 2$$

gives $\quad y = 1, -3.$

Then A is $(3, 1)$, $B(-1, -3)$.

You may reverse the choice of A, B of course.

Thus $\quad AB^2 = (3 + 1)^2 + (1 + 3)^2$
$\quad = 32$

so that $\quad AB = \sqrt{32} = 4\sqrt{2}.$

Example 6.11

Prove that the line $y - x + 3 = 0$ is a tangent to the circle

$$x^2 + y^2 - 2x - 4y - 3 = 0.$$

Our strategy here is to show that the line intersects the circle in one point (or two coincident points).

Substitution of $y = x - 3$ into the equation of the circle gives

$$x^2 + (x-3)^2 - 2x - 4(x-3) - 3 = 0.$$

$\therefore$ $2x^2 - 12x + 18 = 0$

so that $x^2 - 6x + 9 = 0.$

Then $(x-3)^2 = 0$

so that $x = 3$ (twice).

The point of intersection is (3, 0).

or use the quadratic formula

Example 6.12

Find the relation between m and c if $y = mx + c$ is a tangent to the circle

$$x^2 + y^2 = a^2.$$

Substitution of $y = mx + c$ into the equation gives

$$x^2 + (mx + c)^2 = a^2.$$

$\therefore$ $x^2(1 + m^2) + 2mcx + c^2 - a^2 = 0.$

The conditions for this quadratic equation to contain two equal roots is

$$(2mc)^2 = 4(1 + m^2)(c^2 - a^2).$$

$\therefore$ $m^2c^2 = c^2 - a^2 + m^2c^2 - m^2a^2.$

$\therefore$ $c^2 = a^2(1 + m^2)$

or $c = \pm a\sqrt{(1 + m^2)}\,.$

for $ax^2 + bx + c = 0$, $b^2 - 4ac = 0$

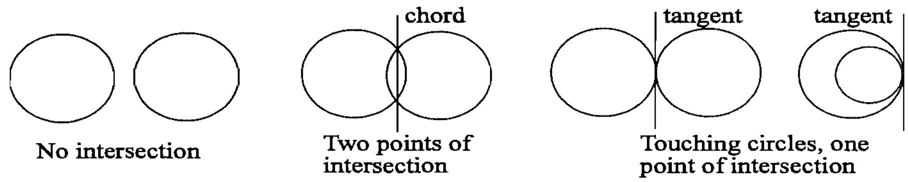
Don't remember the result: remember the condition for equal roots.

Exercises 6.4

1. The line $y = -x + 3$ intersects the circle $x^2 + y^2 - 4x - 2y + 3 = 0$ at the points A and B. Given the point $C(3, 2)$ show that AC and BC are perpendicular. Is C on the circle?
2. Find the length of the chord made by the intersection of the line $x + y = 4$ with the circle $x^2 + y^2 = 25$. Hint : retain the surds.
3. Find the point at which the line $x - 4y - 3 = 0$ touches the circle
$$x^2 + y^2 - 4x - 8y + 3 = 0.$$
4. Find the values of m if $y = mx$ is a tangent to the circle $x^2 + y^2 - 10x + 16 = 0$ and hence find the equations from the origin to the circle.
5. (a) Find a relation between m and c if the line $y = mx + c$ passes through the point (1,2).
   (b) Find a relation between m and c if the line $y = mx + c$ is a tangent to the circle $x^2 + y^2 = 4$.
   (c) Use the results of (a) and (b) to find the equations of the tangents from the point (1, 2) to the circle defined in (a).
6. Find the equations of the tangents of gradient $\frac{3}{4}$ to the circle $x^2 + y^2 = 4$.

The intersection of two circles

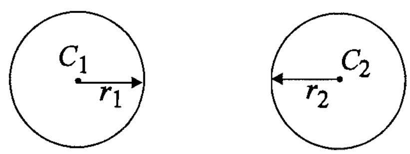
Two circles may or may not intersect. If they intersect they may intersect in one or two points.



When the circles intersect in two points, they have a common chord; when circles intersect in one point, they have a common tangent.

It is easy to check whether circles intersect and, if they do, the number of points of intersection. In the following, the circles have centres C_1, C_2 and radii r_1 and r_2.

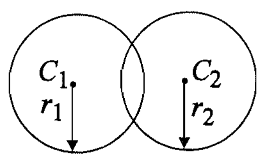
(a) No intersection



In this case, the distance $C_1C_2 > r_1 + r_2$.

The distance between the centres is greater than the sum of radii.

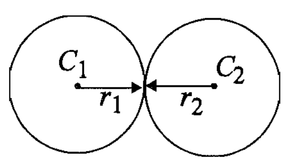
(b) Two points of intersection



In this case, $C_1 C_2 < r_1 + r_2$.

The distance between the centres is less than the sum of radii.

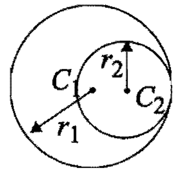
(c) One point of intersection (circles touching externally)



In this case, $C_1 C_2 = r_1 + r_2$.

The distance between the centres equals the sum of the radii.

(d) <u>One point of intersection (circles touching internally)</u>



The distance between the centres = difference of the two radii.

In this case, $C_1 C_2 = r_1 - r_2$.

Example 6.13

Investigate whether the following pairs of circles intersect. Where they do, state the number of points of intersection.

(a) $x^2 + y^2 - 4x - 2y + 1 = 0$; $x^2 + y^2 + 4x - 6y - 12 = 0$

(b) $x^2 + y^2 + 2x = 0$; $x^2 + y^2 - 6x - 4y + 9 = 0$

(c) $x^2 + y^2 = 16$; $5x^2 + 5y^2 - 18x - 24y + 40 = 0$

(d) $x^2 + y^2 + 4x - 4y + 4 = 0$; $x^2 + y^2 - 2x - 4y + 4 = 0$

(a) For the first circle, the centre is (2, 1) and the radius is

$$\sqrt{2^2 + 1^2 - 1} = 2.$$

$(x-2)^2 + (y-1)^2 = 4$

For the second circle, the centre is (–2,3) and the radius is

$$\sqrt{(-2)^2 + 3^2 + 12} = \sqrt{25} = 5.$$

$(x+2)^2 + (y-3)^2 = 25$

The distance between the centres is

$$\sqrt{(-2-2)^2 + (3-1)^2} = \sqrt{4^2 + 2^2} = \sqrt{20}.$$

Now the sum of the radii > distance between centres so that the circles intersect in two points.

$5 + 2 > \sqrt{20}$

(b) For the first circle, the centre is (– 1, 0) and the radius is

$$\sqrt{(-1)^2 + 0^2 - 0} = 1.$$

For the second circle, the centre is (3, 2) and the radius is

$$\sqrt{3^2 + 2^2 - 9} = \sqrt{4} = 2.$$

The distance between the centres is $\sqrt{(3+1)^2 + 2^2} = \sqrt{20}$.

Now the sum of the radii < distance between centres so that the circles do not intersect.

$2 + 1 < \sqrt{20}$

(c) For the first circle, the centre is (0, 0) and the radius is

$$\sqrt{0^2 + 0^2 - (-16)} = 4.$$

For the second circle, the centre is $\left(\frac{9}{5}, \frac{12}{5}\right)$

and the radius is $\sqrt{\left(\frac{9}{5}\right)^2 + \left(\frac{12}{5}\right)^2 - 8} = 1$.

The distance between the centres is $\sqrt{\left(\frac{9}{5} - 0\right)^2 + \left(\frac{12}{5} - 0\right)^2} = 3$.

Now the distance between the centres = difference of the radii so that the circles touch internally.

$3 = 4 - 1$

(d) For the first circle, the centre is (– 2, 2) an the radius is

$$\sqrt{(-2)^2 + (2)^2 - 4} = 2.$$

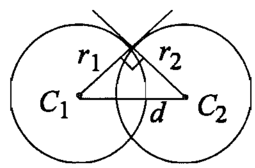
For the second circle, the centre is (1, 2) and the radius is $\sqrt{1^2 + 2^2 - 4} = 1$.

The distance between the centres is $\sqrt{(-2-1)^2 + (2-2)^2} = 3$.

Now the distance between the centres = the sum of the radii so that the circles touch externally.

Orthogonal Circles

If the tangents to two circles at their points of intersection are perpendicular, the circles are said to be orthogonal.



In this case, if the radii are r_1 and r_2 and $C_1C_2 = d$ is the distance between the centres, it follows by Pythagoras' Theorem that

$$d^2 = r_1^2 + r_2^2.$$

Example 6.14

Show that the circles

$$x^2 + y^2 - 2x + 2y - 9 = 0$$
$$x^2 + y^2 + 12x - 3 = 0$$

are orthogonal.

For the first circle, the centre is (1, – 1) and the radius is $\sqrt{11}$.

For the second circle, the centre is (– 6, 0) and the radius is $\sqrt{39}$.

The distance between the centres is

$$\sqrt{(-6-1)^2 + (0-(-1)^2} = \sqrt{50}.$$

Then the sum of the squares of radii is

$$(\sqrt{11})^2 + (\sqrt{39})^2 = 11 + 39 = 50,$$

which is the square of the distance between the centres.

Thus, the circles are orthogonal.

When the circles intersect, it is straightforward, in principle at least, to find the points of intersection.

Example 6.15

Find the points of intersection of the circles

$$x^2 + y^2 = 1, \quad (1)$$
$$x^2 + y^2 + 2x - 4y + 3 = 0. \quad (2)$$

We have to solve these equations simultaneously.

Subtracting (1) from (2), we have

$$2x - 4y + 4 = 0$$

or $$x - 2y + 2 = 0. \quad (3)$$

(3) describes the common chord of the circles. To find the points of intersection of the circles, we find where the common chord intersects one of them.

We substitute from (3) into (1) for x.

From (3), $x = 2y - 2.$

Then (1) becomes,

$$(2y-2)^2 + y^2 = 1$$

so that $$5y^2 - 8y + 3 = 0$$

$\therefore$ $$(5y-3)(y-1) = 0$$

$\therefore$ $$y = \frac{3}{5}, 1$$

or use the quadratic formula

When $y = \frac{3}{5}$, $x = 2 \times \frac{3}{5} - 2 = -\frac{4}{5}$,

$y = 1$, $x = 2 \times 1 - 2 = 0.$

using (3)

The points of intersection are therefore $\left(-\frac{4}{5}, \frac{3}{5}\right)$, and (0, 1).

Exercises 6.5

1. Show, without finding the points of intersection, that the circles
$x^2 + y^2 = 4$ and $x^2 + y^2 - 4x - 2y - 4 = 0$ intersect in two points.

2. Show that the circles
$x^2 + y^2 + 10x - 4y - 3 = 0$ and $x^2 + y^2 - 2x - 6y + 5 = 0$ are orthogonal.

3. Show that the circles
$5x^2 + 5y^2 - 6x - 8y = 0$
$x^2 + y^2 - 6x - 8y + 16 = 0$
touch each other.

4. The circles
$x^2 + y^2 - 6x - 8y + 9 = 0$,
$x^2 + y^2 = 9$,
intersect at two points.
Find the coordinates of the point where the common chord intersects the line joining the centres.

5. Prove that the circles
$x^2 + y^2 + x + 3y = 0$,
and $x^2 + y^2 - 2x - 6y = 0$
touch each other. Find the coordinates of the point of contact and the equation of the common tangent at that point.

6. The circles
$x^2 + y^2 = a^2$
and $x^2 + y^2 - 10x - 24y + 105 = 0$
touch externally at a point. Given that $a > 0$, find the value of a.

Chapter 7

More Differentiation

Differentiation was introduced in **P1.** There, a first principles approach was used to differentiate polynomial functions.
This chapter has two main aims. First, it develops some techniques of differentiation to supplement the first principles approach.
Secondly, it considers the differentiation of some additional functions.
Note that in this chapter we abuse notation by referring to f(x) as a function.

7.1 Differentiating composite functions (function of a function rule)

It is essential, before considering the differentiation of composite functions, to recognise composite functions when they occur.

Example 7.1
Identify the composite functions in the following.

(i) $(x+2)^2$	(ii) $\sin 3x$	(iii) $x(x+2)$
(iv) $\cos(x^2+2)$	(v) $x \sin x$	(vi) $\dfrac{x^2}{x+2}$
(vii) $x2^x$	(viii) $\sqrt{1+x}$	(ix) 3^{x+5}

Here we attempt to put each function in the form f(g(x)) where f and g are functions to be identified.

(i) Composite. $(x+2)^2$ may be considered as f(g(x)) where g(x) = $x + 2$ and f(x) = x^2.

(ii) Composite. Sin $3x$ may be considered as f(g(x)) where g(x) = $3x$ and f(x) = $\sin x$.

(iii) Not composite. $x(x+2)$ is not of the form f(g(x)) but is of the form f(x) × g(x) where f(x) = x, g(x) = $x + 2$.

(iv) Composite. Cos(x^2 + 2) is of the form f(g(x)) where g(x) = $x^2 + 2$ and f(x) = $\cos x$.

(v) Not composite. $x \sin x$ is the product of f(x) = x, g(x) = $\sin x$.

(vi) Not composite. $\dfrac{x^2}{x+2}$ is the quotient of f(x) = x^2 and g(x) = $x + 2$.

(vii) Not composite. Product of f(x) = x, g(x) = 2^x.

(viii) Composite with g(x) = $x + 1$, f(x) = $\sqrt{x}$.

(ix) Composite with g(x) = 3^x, f(x) = $x + 5$.

Note that $\sin 3x$ is not $\sin \times 3x$

$\cos(x^2+2)$ is not $\cos \times (x^2+2)$

Exercises 7.1

Identify the following as composite or non-composite functions. In the case of the composite functions, identify the inner and outer functions (g(x) and f(x) respectively).

(i) $x\sin 3x$	(ii) $\sqrt{x^3+2x+1}$	(iii) $\tan(5x+7)$
(iv) $(x^2+3)(x^2+5)$	(v) $(x^2+3)^{5/3}$	(vi) $\dfrac{x}{\sin 4x}$
(vii) $(x+3)^2+5$	(viii) 6^x+7	(ix) $(x+3)\cos x$

We obtain insight into the differentiation of composite functions by the following examples.

Example 7.2

Differentiate the following by first multiplying out the brackets. For convenience we label all the functions as k(x).

(i) $k(x) = (3x+2)^2$ (ii) $k(x) = (x^2 +.1)^2$ (iii) $k(x) = (7x^2-2)^2$.

(i) Now $k(x) = 9x^2 + 12x + 4$

so $k'(x) = 18x + 12$

$= 6(3x+2) = 2.3(3x+2)$.

$(a+b)^2 = a^2 + 2ab + b^2$

The reason for factorising will become clear.

(ii) Now $k(x) = x^4 + 2x^2 + 1$

so $k'(x) = 4x^3 + 4x = 4x(x^2+1) = 2.2x(x^2+1)$.

(iii) Now $k(x) = 49x^4 - 28x^2 + 4$

so $k'(x) = 196x^3 - 56x$

$= 28x(7x^2-2) = 2.14x(7x^2-2)$.

We summarise the results below.

Function	Derived Function
$(3x+2)^2$	$2(3x+2).3$
$(x^2+1)^2$	$2(x^2+1).2x$
$(7x^2-2)^2$	$2(7x^2-2).14x$

The reason for the reordering of the factors will soon be apparent.

In each case,

$k(x) = (\text{expression})^2$

and $k'(x) = 2(\text{expression}) \times \text{derivative of expression}$:-

$$2\begin{array}{|c|}\hline 3x+2 \\ x^2+1 \\ 7x^2-2 \\ \hline\end{array} \times \begin{array}{|c|}\hline 3 \\ 2x \\ 14x \\ \hline\end{array}$$

The general rule suggested by the above examples is valid.

Exercises 7.2
Check that the rule
'if $k(x) = (\text{expression})^2$, then $k'(x) = 2(\text{expression}) \times$ derivative of expression'
holds in the following cases.
(i) $k(x) = (2x-3)^2$ (ii) $k(x) = (3x^2+4)^2$ (iii) $k(x) = (x^3+x)^2$.

A similar rule applies for different powers of the expression.

Exercises 7.3
Check that if

$$k(x) = (\text{expression})^3$$

then $k'(x) = 3(\text{expression})^2 \times$ derivative of expression

for (i) $k(x) = (x+1)^3$
(ii) $k(x) = (2x-1)^3$
(iii) $k(x) = (x^2+1)^3$

$(x+1)^3 = x^3+3x^2+3x+1$
$(2x-1)^3 = 8x^3-12x^2+6x-1$
$(x^2+1)^3 = x^6+3x^4+3x^2+1$

The generalisation of the results considered in exercises 7.2 and 7.3 is the following :-

Rule (I)

If $k(x) = (\text{expression})^n$ then $k'(x) = n(\text{expression})^{n-1} \times$ derivative of expression. This result applies in fact if n is a positive, negative integer or rational number.

Example 7.3
Use rule I to write down the derivatives of
(i) $(5x+6)^9$ (ii) $(2x-1)^{-1}$ (iii) $(2x^3+x^2-4)^{-3/2}$

(i) If $k(x) = (5x+6)^9$
then

$$k'(x) = 9(5x+6)^{9-1} \times (5)) \longleftarrow \text{derivative of } 5x+6$$
$$= 45(5x+6)^8,$$

where for convenience we group factors finally.

(ii) For $(2x-1)^{-1}$, expression $= 2x-1$ and $n = -1$.
Then derivative is $-1(2x-1)^{-1-1} \times (2) \longleftarrow$ derivative of $2x-1$

$$= -(2x-1)^{-2} \times 2$$
$$= -\frac{2}{(2x-1)^2}.$$

(iii) Expression $= 2x^3+x^2-4$, $n = -\dfrac{3}{2}$.

Derivative

$$= -\frac{3}{2}(2x^3+x^2-4)^{-3/2-1} \times (6x^2+2x) \longleftarrow \text{derivative of } 2x^3+x^2-4$$
$$= -3x(3x+1)(2x^3+x^2-4)^{-5/2}$$
$$= \frac{-3x(3x+1)}{(2x^3+x^2-4)^{5/2}}.$$

Note that $-\frac{3}{2}-1 = -\frac{5}{2}$ not $-\frac{1}{2}$

Exercises 7.4

1 Use Rule I to write down the derived functions in the following cases :-

(i) $(9x-2)^4$ (ii) $(3x^2+2)^{-1}$ (iii) $(x^2+3x+4)^2$

(iv) $(2x+1)^{1/2}$ (v) $(x^7+4x^3)^3$ (vi) $\dfrac{1}{x+1}$

(vii) $(x^2-4x+2)^{-5/2}$ (viii) $\dfrac{1}{(3x+2)^{1/2}}$ (ix) $\left(x+\dfrac{1}{x}\right)^4$

(x) $\left(x^2+\dfrac{1}{x}\right)^{\frac{1}{2}}$ (xi) $\left(3x+\dfrac{1}{x}+\dfrac{1}{x^2}\right)^{\frac{1}{2}}$ (xii) $\left(7x-\dfrac{4}{x}\right)^{-\frac{1}{2}}$

Rule I may be rewritten as follows :-

let $y = (g(x))^n$

so that $y = f(g(x))$,

where $f(x) = x^n$.

Then $\dfrac{dy}{dx} = n(g(x))^{n-1}g'(x)$

can be written as

$$\frac{dy}{dx} = f'(g(x))\, g'(x).$$

Thus Rule (I')

g(x) was called 'expression' previously

$f(x) = x^n$
$f'(x) = nx^{n-1}$
$f'(g(x)) = n(g(x))^{n-1}$

If	$y = f(g(x))$
then	$\dfrac{dy}{dx} = f'(g(x)) \times g'(x)$

The significance of this rewriting will become apparent later.

I' is a general rule whatever the functions f and g. The rule, of course, concerns the differentiation of composite functions and is often referred to as the **function of a function rule**. We shall obtain practice in the use of this rule with other functions later.

For the moment, we confine the discussion here to giving a proof of (I'), followed by some further examples. The proof of I' is non examinable.

Let $y = f(g(x))$.

If $u = g(x)$ then

$y = f(u)$.

If δx is a small increase in x and δu, δy are corresponding small increases in u and y, respectively, then

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x}.$$

Then as $\delta x \to 0$,

$$\frac{dy}{dx} = \lim_{\delta x \to 0}\frac{\delta y}{dx} = \lim_{\delta x \to 0}\left(\frac{\delta y}{du} \times \frac{\delta u}{dx}\right)$$

$$= \lim_{\delta u \to 0}\frac{\delta y}{du}\ \lim_{\delta x \to 0}\frac{\delta u}{dx}$$

$$= \frac{dy}{du} \times \frac{du}{dx}$$

We assume that the limit of the product is the product of the limits, a non-trivial result, incidentally.

$$= f'(u) \times g'(x)$$
$$= f'(g(x)) \times g'(x).$$

$y = f(u)$, $u = g(x)$

This establishes Rule I'.

If	y	$= f(g(x))$
then	$\dfrac{dy}{dx}$	$= f'(g(x))\, g'(x).$

The method of proof given above suggests another method of representing the differentiation of composite functions.

Example 7.4

(i) If $y = (x^3 + 3x^2 + 1)^{\frac{3}{2}}$ we may write $y = u^{\frac{3}{2}}$ where $u = x^3 + 3x^2 + 1$.

Then
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\downarrow \quad \downarrow$$
$$= \frac{3}{2}u^{\frac{1}{2}} \times (3x^2 + 6x)$$
$$= \frac{9}{2}x(x+2)(x^3 + 3x^2 + 1)^{\frac{1}{2}},$$
on factorising and restoring the function $u = x^3 + 3x^2 + 1$.

(ii) If $y = \left(x^4 + 6x^2 - \dfrac{2}{x}\right)^{-\frac{1}{2}}$

then $y = u^{-\frac{1}{2}}$ where $u = x^4 + 6x^2 - \dfrac{2}{x}$.

Now $\dfrac{dy}{du} = -\dfrac{1}{2}u^{-\frac{1}{2}-1} = -\dfrac{1}{2}u^{-\frac{3}{2}}$ (from **P1**)

and $\dfrac{du}{dx} = 4x^3 + 12x - 2(-1)x^{-1-1}$

$$= 4x^3 + 12x + \frac{2}{x^2}.$$

$-\dfrac{2}{x} = -2x^{-1}$

Thus
$$\frac{dy}{dx} = -\frac{1}{2}\left(x^4 + 6x^2 - \frac{2}{x}\right)^{-\frac{3}{2}} \times \left(4x^3 + 12x + \frac{2}{x^2}\right)$$
$$= -\left(2x^3 + 6x + \frac{1}{x^2}\right)\left(x^4 + 6x^2 - \frac{2}{x}\right)^{-\frac{3}{2}},$$
on restoring the function $u = x^4 + 6x^2 - \dfrac{2}{x}$ and dividing by 2.

Exercises 7.5

Find $\dfrac{dy}{dx}$ in the following cases :-

(i) $y = x^{-4}$ (ii) $y = \left(x + \dfrac{1}{x}\right)^{12}$ (iii) $y = (3x^2 + 5x - 61)^{\frac{5}{2}}$

(iv) $y = \left(9x^4 - 7x^3 - \dfrac{3}{x^2}\right)^{-\frac{5}{2}}$ (v) $y = \dfrac{1}{7x^9 - 3x^6 + 2x + 1}$

(vi) $y = \dfrac{1}{(2x^7 - 7x^2 + 1)^{\frac{3}{2}}}$ (vii) $y = \dfrac{1}{\sqrt{3x^2 + 5x - \frac{1}{x^3}}}$

(viii) $y = \left(\sqrt{x} + \dfrac{2}{\sqrt{x}} + 3\right)^{-6}$.

We now leave differentiation of the composite function briefly and consider the differentiation of the so-called exponential function.

7.2 The function f(x) = e^x and its derived function

Functions such as 2^x, $(\frac{1}{2})^x$ were discussed in Chapter 5. The functions are special cases of the general function $f(x) = a^x$ where $a > 0$.

To differentiate $f(x) = a^x$ we may proceed as follows.

This is non examinable.

We let $y = a^x$ and δx, δy be corresponding small increments in x and y respectively.

Then $f'(x) = \dfrac{dy}{dx} = \lim \dfrac{\delta y}{\delta x}$.

as defined in P1

Now $y = a^x$

and $y + \delta y = a^{x+\delta x}$.

Then $\delta y = a^{x+\delta x} - a^x$

so $\dfrac{\delta y}{\delta x} = \dfrac{a^{x+\delta x} - a^x}{\delta x}$

and
$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{a^{x+\delta x} - a^x}{\delta x} = \lim_{\delta x \to 0} \frac{a^x\left(a^{\delta x} - 1\right)}{\delta x}.$$

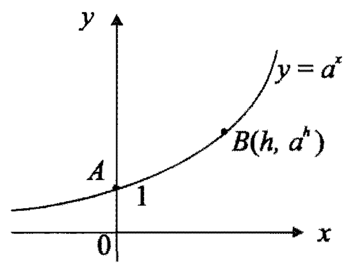
Alternatively,
$f(x) = a^x$
$f(x + h) = a^{x+h}$
$f'(x) = \lim\limits_{h \to 0} \dfrac{a^{x+h} - a^x}{h}$.

so $f'(x) = \lim\limits_{h \to 0} \dfrac{a^x(a^h - 1)}{h}$
$= a^x \lim\limits_{h \to 0} \left(\dfrac{a^h - 1}{h}\right)$

We note that $\lim\limits_{\delta x \to 0} \dfrac{\left(a^{\delta x} - 1\right)}{\delta x}$ and $\lim\limits_{h \to 0} \dfrac{\left(a^h - 1\right)}{h}$ are expressions for the same limit.

The form $\lim_{h \to 0} \frac{\left(a^h - 1\right)}{h}$ will be considered here.

The value of the limit is dependent on a. To make progress, we consider the graph of $y = a^x$ where $a \geq 1$. The graph is as shown.



All the curves of the form $y = a^x$ pass through the point (0, 1).

The points $A(0, 1)$ and $B(h, a^h)$ are shown on the graph.
The slope of the chord AB is then

$$\frac{\text{difference of } y\text{'s}}{\text{difference of } x\text{'s}} = \frac{a^h - 1}{h - 0} = \frac{a^h - 1}{h}.$$

As $h \to 0$, the slope of the chord tends to the slope of the tangent at A. In other words,

$$\text{slope of tangent at } A = \lim_{h \to 0} \frac{a^h - 1}{h}.$$

We investigate this limit for various values of a by means of a calculator. The y^x button on the calculator should be used. The results are quoted correct to 2 decimal places.

a	a^x	Values of $\frac{a^h-1}{h}$: $h = 0.1$	$h = 0.01$	$h = 0.001$	Approximate limit
1	$1^x = 1$	0	0	0	0
2	2^x	0.72	0.70	0.69	0.69
3	3^x	1.16	1.10	1.10	1.10
4	4^x	1.49	1.40	1.39	1.39

In passing it should be noted that when $a = 1$, $a^h = 1^h = 1$ for all values of h so

$$\frac{a^h - 1}{h} = \frac{1^h - 1}{1} = \frac{0}{1} = 0.$$

The last column in the table gives approximate values of $\lim_{h \to 0} \frac{a^h - 1}{h}$ for various values of a.

Recalling that if $f(x) = a^x$, we see that since

$$f'(x) = a^x \lim_{h \to 0} \frac{a^h - 1}{h},$$

the approximate derivatives for 1^x, 2^x, 3^x, 4^x are as shown in the following table.

$f(x)$	$f'(x)$ (approximately)
1^x	0
2^x	0.69×2^x
3^x	1.10×3^x
4^x	1.39×4^x

Table 1

Except for 1^x the derivatives are the original functions multiplied by (different) non-zero constants.

In the table, the coefficients are given correct to 2 decimal places. The values of the coefficients for other values of a can be found by means of a calculator.

Exercises 7.6

1 Show that $\lim_{h \to 0} \frac{2.7^h - 1}{h} \approx 0.99$,

and hence write down an approximate derivative for $f(x) = 2.7^x$.

2 Show that $\lim_{h \to 0} \frac{2.72^h - 1}{h} \approx 1.001$

and hence write down an approximate derivative for $f(x) = 2.72^x$.

We deduce from Table 1 and the solutions of Exercises 7.6 that $\lim_{h \to 0} \frac{a^h - 1}{h} = 1$ for some value of a between 2 and 3. In fact from the solutions to Exercises 7.6 this value of a is between 2.7 and 2.72. This number is called e in mathematics and has approximate value 2.718282, correct to 6 decimal places.

The significance of e is that if $f(x) = e^x$

then $f'(x) = 1 \times e^x = e^x$.

This is a remarkable result and bears repeating :-

the derived function of $f(x) = e^x$

is $f'(x) = e^x$, the same function.

Rule (II)

If	$f(x) = e^x$
then	$f'(x) = e^x$.

The function e^x is of fundamental importance in mathematics due to the fact that it is unaltered by differentiation.

Being a function, e^x can be used in the same way as other functions. In particular, it can be involved in the composition of functions.

Example 7.5

Differentiate the functions (i) e^{2x+1} (ii) e^{x^2+4x+2}

(i) The function is f(g(x)) where g(x) = 2x + 1 and f(x) = e^x.

Then its derivative f '(g(x)) × g'(x)

$$= e^{2x+1} \times 2$$
$$= 2e^{2x+1}.$$

$f(x) = e^x$
$f'(x) = e^x$
$f'(g(x)) = e^{2x+1}$

Alternatively,

$$y = e^u \quad \text{where } u = 2x + 1.$$

Then
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= e^u \times 2$$
$$= 2e^{2x+1}.$$

Note that if $\frac{d}{dx}(e^x) = e^x$ then $\frac{d}{du}(e^u) = e^u$.

(ii) The function is f(g(x)) where g(x) = $x^2 + 4x + 2$ and f(x) = e^x.

Then the derivative is f '(g(x)) × g'(x)

$$= e^{x^2+4x+2} \times (2x + 4)$$
$$= (2x + 4)e^{x^2+4x+2}.$$

or if $y = e^u$ where $u = x^2 + 4x + 2$,

then
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= e^u \times (2x + 4)$$
$$= (2x + 4)e^{x^2+4x+2}.$$

It is useful to streamline the differentiation of functions such as $e^{g(x)}$.

If $y = e^{g(x)}$

then $y = e^u$ where $u = g(x)$.

Thus
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= e^u \times g'(x)$$
$$= e^{g(x)} \times g'(x).$$

$\frac{dy}{dx} = f'(g(x)g'(x)$
$= e^{g(x)} g'(x)$

Rule (III)

If $y = e^{g(x)}$ then $\frac{dy}{dx} = e^{g(x)} g'(x).$

The result, therefore, of differentiating $e^{(\text{expression})}$ is

$e^{\text{expression}}$ × derivative of expression.

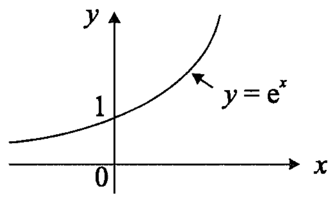
Exercises 7.7
The following examples make use of Rule III.
Find the derived functions of

(i) e^{3x} (ii) e^{x^2} (iii) e^{x^3+2} (iv) $e^{x+\frac{1}{x}}$

(v) e^{-x} (vi) e^{-4x} (vii) e^{x^3-x+1}

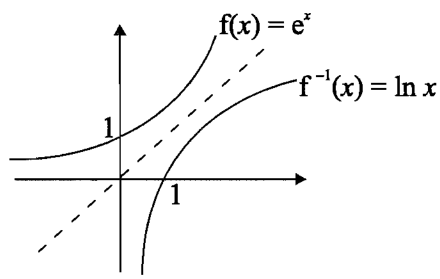
7.3 Log$_e$ x and its derived function

The function $f(x) = e^x$ has derived function $f'(x) = e^x$. This derived function is positive for all values of x and the function $f(x)$ is therefore an increasing function. The graph of $f(x) = e^x$ is as shown.



The function f is one-one and has an inverse f^{-1}.
We define the inverse of f (given by $f(x) = e^x$) to be $f^{-1}(x) = \ln x$ or $\log_e x$.
This inverse function is called the **logarithmic function to base e**.
The graphs of $f(x) = e^x$ and $f^{-1}(x) = \ln x$ are shown below.

In Chapter 5 the inverse of $f(x) = 10^x$ was $f^{-1}(x) = \log_{10} x$, the logarithmic function to base 10.



The graph of f^{-1} is the reflection of the graph of f in the line $y = x$.
Now, by definition, the action of f (or f^{-1}) reverses the effect of f^{-1} (or f).

Thus $\quad f f^{-1}(x) = x$

or $\quad f^{-1}f(x) = x$.

Since $f(x) = e^x$ and $f^{-1}(x) = \ln x$, we have:-

Rule (IV)

$$e^{\ln(x)} = x, \quad (1)$$
$$\ln(e^x) = x. \quad (2)$$

The logarithmic function has a number of other properties which will not be discussed here. We confine our discussion to the differentiation of ln x.
The result (1) in Rule IV, taken with Rule III above, may be used to find the derived function of ln x.
Suppose $g(x) = \ln x$. Then (1) above may be written as

$$e^{g(x)} = x. \quad (3)$$

Differentiate both sides of (3) with respect to x.
The right hand side, i.e., x, becomes 1 when differentiated. What about the differentiation of $e^{g(x)}$?
By rule III, $e^{g(x)}$ becomes $e^{g(x)} \times g'(x)$ when differentiated.

Thus if $\quad e^{g(x)} = x$

then $\quad e^{g(x)} \times g'(x) = 1$

so $\quad g'(x) = \dfrac{1}{e^{g(x)}}$.

Recalling form (3) that $e^{g(x)} = x$,

we have $\quad g'(x) = \dfrac{1}{x}$.

Thus Rule (V)

If $g(x) = \ln x$ then $g'(x) = \dfrac{1}{x}$.

$y = \ln x$
$\dfrac{dy}{dx} = \dfrac{1}{x}$

The function ln x may be composed with other functions e.g. $\ln(3x^2 + 2)$, $3\ln(x + 2)$, $\ln(e^{3x} + x^2 - 5)$ and so on.
Such functions may be differentiated using Rule I'.

Example 7.6
Differentiate the following functions :

(i) $\ln 2x$ (ii) $\ln x^2$ (iii) $\ln\left(x - \dfrac{1}{x}\right)$ (iv) $(\ln x)^2$ (v) $\ln\left(\dfrac{1}{x}\right)$

(i) Now $\ln 2x = f(g(x))$,

where $f(x) = \ln x$, $f'(x) = \dfrac{1}{x}$

$g(x) = 2x$, $g'(x) = 2$

and $f'(g(x)) = \dfrac{1}{g(x)} = \dfrac{1}{2x}$..

Then if $y = \ln 2x$,

$$\frac{dy}{dx} = f'(g(x))\,g'(x) = \frac{1}{2x} \times 2 = \frac{1}{x}.$$

Alternatively,

if $y = \ln 2x$,

then $y = \ln u$ where $u = 2x$.

Thus, $$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{1}{u} \times 2 = \frac{1}{2x} \times 2 = \frac{1}{x}.$$

$f'(x) = \dfrac{1}{x}$
$f'(g(x)) = \dfrac{1}{g(x)}$

It is no coincidence that ln $2x$ and ln x have the same derivative

(ii) Using the alternative approach,

$$y = \ln u \text{ where } u = x^2$$

and
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= \frac{1}{u} \times 2x = \frac{1}{x^2} \times 2x = \frac{2}{x}.$$

(iii) $y = \ln u$ where $u = x - \frac{1}{x}$

and
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= \frac{1}{u} \times \left(1 + \frac{1}{x^2}\right)$$
$$= \frac{1 + \frac{1}{x^2}}{x - \frac{1}{x}}$$
$$= \frac{x^2 + 1}{x^3 - x},$$

on multiplying top and bottom by x^2.

$-\frac{1}{x} = -x^{-1}$
differentiated gives
$(-1)(-x^{-1-1}) = \frac{1}{x^2}$

(iv) $y = u^2$ where $u = \ln x$.

Then
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 2u \times \frac{1}{x} = 2(\ln x) \times \frac{1}{x}$$
$$= \frac{2}{x}\ln x.$$

(v) $y = \ln u$ where $u = \frac{1}{x}$.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= \frac{1}{u} \times -\frac{1}{x^2}$$
$$= \frac{1}{\frac{1}{x}} \times -\frac{1}{x^2} = -\frac{1}{x}.$$

$u = x^{-1}$
$\frac{du}{dx} = (-1).x^{-1-1}$
$= -\frac{1}{x^2}$

Differentiation of functions of the form $\ln(g(x))$ may be streamlined.

If $y = \ln(g(x))$

so $y = \ln u$ where $u = g(x)$.

Then
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= \frac{1}{u} \times g'(x) = \frac{g'(x)}{g(x)}.$$

Rule (VI)

Thus if
$y = \ln(g(x))$
$\dfrac{dy}{dx} = \dfrac{g'(x)}{g(x)}$.

Then we may differentiate $\ln(x^2 + 3x + 4)$ immediately to obtain

$$\frac{2x+3}{x^2+3x+4}\left(\frac{g'(x)}{g(x)}\right).$$

Exercises 7.8

The following exercises make use of Rules I – VI. You may assume that where necessary sums of terms may be differentiated term by term.

1. <u>Write down</u> by means of Rule VI the derived functions of the following :-
   (i) $\ln(5x)$ (ii) $\ln(6x + 5)$ (iii) $\ln(x^2 + x)$
   (iv) $\ln\left(\dfrac{1}{x^2}\right)$ (v) $\ln(9x^2 + 4x + 3)$ (vi) $\ln\left(x^2 + \dfrac{1}{x}\right)$
   (vii) $\ln(x^7 + 1)$ (viii) $\ln x^{-\frac{5}{2}}$ (ix) $\ln((x + 1)^2)$
   (x) $\ln((x^2 + x)^3)$ (xi) $\ln(e^{2x})$ (xii) $\ln(e^{x+5})$

2. Differentiate
   (i) $e^{\ln(x^2+1)}$ (ii) $(\ln x)^3$ (iii) $e^{\ln x^3}$ (iv) $e^{3\ln x}$

3. Find the derived functions of
   $f(x) = \ln(x^3)$, $g(x) = \ln(x^4)$, $h(x) = \ln(x^7)$
   and show that $h'(x) = f'(x) + g'(x)$.

4. Show that if $f(x) = \ln(x^n)$ then $f'(x) = \dfrac{n}{x}$.

5. Show that if $k(x) = \ln(x^9)$, $m(x) = \ln(x^6)$, $n(x) = \ln(x^3)$
   then $n'(x) = k'(x) - m'(x)$.

6. Show that if $f(x) = \ln(e^x)$ then $f'(x) = 1$.
   Which other function has derived function 1?
   How do you reconcile the results? (Hint : see Rule IV, (2))

7. Show that if $f(x) = e^{\ln x}$ then $f'(x) = 1$.
   Explain the result.

7.4 More techniques of differentiation

In **P1** it was pointed out that a function consisting of the algebraic sum of multiples of powers of x and constants may be differentiated term by term.
In fact, sums of any type of functions may be differentiated term by term. Thus, if

$$\text{m}(x) = x^4 + 9x^3 + e^x + \ln x$$

then $$\text{m}'(x) = 4x^3 + 27x^2 + e^x + \frac{1}{x}.$$

Term by term differentiation is justified as follows.

For convenience we consider the sum of three functions. The proof carries over to any finite sum.

Suppose $y = \text{f}(x) + \text{g}(x) + \text{h}(x)$. (1)
Let δx, δy be corresponding small increments in x and y respectively.
Then $y + \delta y = \text{f}(x + \delta x) + \text{g}(x + \delta x) + \text{h}(x + \delta x)$. (2)
Subtract (1) from (2) and group terms in f, g, h on the right hand side.

$$\therefore \quad \delta y = \text{f}(x + \delta x) - \text{f}(x) + \text{g}(x + \delta x) - \text{g}(x) + \text{h}(x + \delta x) - \text{h}(x)$$

and $$\frac{\delta y}{\delta x} = \frac{\text{f}(x+\delta x) - \text{f}(x)}{\delta x} + \frac{\text{g}(x+\delta x) - \text{g}(x)}{\delta x} + \frac{\text{h}(x+\delta x) - \text{h}(x)}{\delta x}$$

Thus $$\frac{\text{d}y}{\text{d}x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \left(\frac{\text{f}(x+\delta x) - \text{f}(x)}{\delta x} + \frac{\text{g}(x+\delta x) - \text{g}(x)}{\delta x} + \frac{\text{h}(x+\delta x) - \text{h}(x)}{\delta x} \right)$$

$$\downarrow$$

$$= \quad \text{f}'(x) \quad + \quad \text{g}'(x) + \quad \text{h}'(x),$$

assuming that the limit of a sum of terms is the sum of the separate limits.

Rule (VII)

Thus the derived function of $\text{f}(x) + \text{g}(x) + \text{h}(x)$ is $\text{f}'(x) + \text{g}'(x) + \text{h}'(x)$, a result which generalises to the sum of any finite number of functions.

We are therefore justified in differentiating

$$e^x + \ln(x^2 + 1) + x^2 + 3x + 5$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

as $$e^x + \frac{2x}{x^2+1} + 2x + 3 \ (+0)$$

Exercises 7.9

Differentiate the following functions :-

(i) $\ln x + e^x$ (ii) $\ln(x^2 + 1) + x^2 + 1$ (iii) $e^{3x} + x^4 + 2$

(iv) $\ln(x^2 + x) + e^{3x-7} + 2$ (v) $\ln(e^x + x)$ (vi) $\ln(e^{x^2} + x)$

(vii) $\ln(3x^2 + 2) + (x - 5)^2$ (viii) $\ln\left(e^x + \frac{1}{x} + 2\right)$ (ix) $\ln(e^x + e^{-x})$

(x) $e^{3\ln x + x^2}$ (xi) $(e^x - x + 2)^4$

Whilst the techniques introduced so far have been useful, there are problems which cannot be treated by these techniques. For example, how do we differentiate,

(i) x^3e^x or (ii) $x^7 \ln (x^2 + 1)$?

Now we are able to differentiate x^3, e^x, x^7, $\ln (x^2 + 1)$ as separate terms but does this enable us to differentiate (i) and (ii)? In fact, it does, but we require another rule.

Can you?

We note that x^3e^x and $x^7 \ln (x^2 + 1)$ are neither sums of functions nor compositions of functions. In fact they are both of the form $f(x) \times g(x)$, i.e. products of functions. The appropriate rule for differentiating such cases is, not surprisingly, known as the **Product Rule**.

(i) $f(x) = x^3$
$g(x) = e^x$, say
(ii) $f(x) = x^7$,
$g(x) = \ln(x^2+1)$
or vice versa

The rule is

Rule (VIII)

$$\frac{d}{dx}(f(x)g(x)) = g(x)f'(x) + f(x)g'(x).$$

Thus to differentiate a product of two functions : we differentiate the first and leave the second alone, then differentiate the second and leave the first alone, and add the two components so obtained.

N.B. The rule is <u>not</u> $\frac{d}{dx} f(x)g(x) = f'(x)g'(x)$

Example 7.7

Differentiate (i) x^3e^x (ii) $x^7 \ln x$ (iii) $x^2 (x + 1)^{20}$

(i)
$$f(x) = x^3, \quad g(x) = e^x,$$
$$f'(x) = 3x^2, \quad g'(x) = e^x,$$
so
$$\frac{d}{dx}(x^3e^x) = e^x.3x^2 + x^3.e^x$$
$$= x^2e^x(3 + x), \text{ on taking out common factors.}$$

(ii)
$$f(x) = x^7, \quad g(x) = \ln x,$$
$$f'(x) = 7x^6, \quad g'(x) = \frac{1}{x},$$
so
$$\frac{d}{dx}(x^7 \ln x) = (\ln x).7x^6 + x^7.\frac{1}{x}$$
$$= 7x^6 \ln x + x^6$$
$$= x^6(7 \ln x + 1).$$

(iii)
$$f(x) = x^2, \quad g(x) = (x + 1)^{20},$$
$$f'(x) = 2x, \quad g'(x) = 20(x + 1)^{19},$$
so
$$\frac{d}{dx}(x^2 (x + 1)^{20}) = (x + 1)^{20}.2x + x^2.20(x + 1)^{19}$$
$$= x(x + 1)^{19} [2(x + 1) + 20x]$$
$$= x(x + 1)^{19} [22x + 2]$$
$$= 2x(x + 1)^{19} [11x + 1].$$

Notice that the function of function rule is used to find $g'(x)$.

For completeness, we give the proof of Rule VIII and give an alternative form of the rule.

Let $\quad y = \mathrm{f}(x)\mathrm{g}(x)$

or $\quad y = uv, \quad (1)$

where $\quad u \equiv \mathrm{f}(x), \quad v \equiv \mathrm{g}(x).$

Now let δx, δu, δv, δy be corresponding small increments in x and y respectively.

Then $\quad y + \delta y = (u + \delta u)(v + \delta v). \quad (2)$

Subtracting (1) from (2), we obtain

$$\begin{aligned} \delta y &= (u + \delta u)(v + \delta v) - uv \\ &= u\,\delta v + v\,\delta u + \delta u\,\delta v. \end{aligned}$$

$$\therefore \quad \frac{\delta y}{\delta x} = u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \frac{\delta u\,\delta v}{\delta x}.$$

When $\delta x \to 0$, $\quad \dfrac{\delta u}{\delta x} \to \dfrac{\mathrm{d}u}{\mathrm{d}x}$

$\quad \dfrac{\delta v}{\delta x} \to \dfrac{\mathrm{d}v}{\mathrm{d}x}$

and $\quad \dfrac{\delta u \delta v}{\delta x} \to 0.$

$$\text{Then} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x \to 0}\left(u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \frac{\delta u\,\delta v}{\delta x}\right)$$

$$= u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x},$$

where $u = \mathrm{f}(x)$, $\dfrac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{g}'(x)$, $v = \mathrm{g}(x)$, $\dfrac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{f}'(x)$.

Thus Rule VIII may be written as

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = v\frac{\mathrm{d}u}{\mathrm{d}x} + u\frac{\mathrm{d}v}{\mathrm{d}x} \qquad \text{(a)}$$

or as given previously,

$$\frac{\mathrm{d}y}{\mathrm{d}x}(\mathrm{f}(x)\mathrm{g}(x)) = \mathrm{g}(x)\mathrm{f}'(x) + \mathrm{f}(x)\mathrm{g}'(x). \qquad \text{(b)}$$

The form VIII (a) is usually the more popular form with students and is often remembered as

$$\mathrm{d}(uv) = v\mathrm{d}u + u\mathrm{d}v$$

or in words

' dee *uv* equals *v* dee *u* plus *u* dee *v* '.

Either form may be used in practice, of course.

Example 7.8

Differentiate (i) $x^2(2x + 1)^3$ (ii) $(x^2 + 1) \ln x$ (iii) $\mathrm{e}^{2x} \ln (3x^4 + x + 1)$.

(i) $u = x^2, \qquad v = (2x + 1)^3,$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2x, \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = 3(2x + 1)^2 . 2 = 6(2x + 1)^2.$$

$\mathrm{f}(x) = x^2$, $\mathrm{g}(x) = (2x + 1)^3$
$\mathrm{f}'(x) = 2x$, $\mathrm{g}'(x) = 6(2x + 1)^2$

[Note in passing the use of the function of a function rule to differentiate $(2x + 1)^3$ or $(\text{expression})^3$].

Then the result is

$$(2x+1)^3.2x + x^2.6(2x+1)^2 = 2x(2x+1)^2[2x+1+3x] \quad \text{(factorising)}$$
$$= 2x(2x+1)^2(5x+1)$$
$$\text{or } 2x(5x+1)(2x+1)^2.$$

(ii) $f(x) = x^2 + 1$, $\quad g(x) = \ln x$,

$f'(x) = 2x$, $\quad g'(x) = \dfrac{1}{x}$.

$u = x^2 + 1$, $v = \ln x$
$\frac{du}{dx} = 2x$, $\frac{dv}{dx} = \frac{1}{x}$

The result is

$$\ln(x).\,2x + (x^2+1).\frac{1}{x} = 2x\ln x + \frac{1}{x}(x^2+1)$$
$$= x(2\ln x + 1) + \frac{1}{x}.$$

Use brackets to distinguish between terms.

(iii) $u = e^{2x}$, $\quad v = \ln(3x^4 + x + 1)$

$\dfrac{du}{dx} = 2e^{2x}$, $\quad \dfrac{dv}{dx} = \dfrac{12x^3+1}{3x^4+x+1}$.

Use Rules III and VI to differentiate e^{2x} and $\ln(3x^4+x+1)$

The result is

$$\ln(3x^4+x+1).2e^{2x} + e^{2x}.\frac{12x^3+1}{3x^4+x+1}$$
$$= e^{2x}\left[2\ln(3x^4+x+1) + \frac{12x^3+1}{3x^4+x+1}\right].$$

One further technique must be considered : the quotient rule. The quotient rule is concerned with the differentiation of functions such as

(i) $\dfrac{x}{x^2+1}$ $\quad$ (ii) $\dfrac{\ln(x^3+4)}{3x+1}$

i.e. the functions of the form

$$\frac{f(x)}{g(x)} \text{ or } \left(\frac{u}{v}\right).$$

We state and use the quotient rule before proving it.

Rule (IX)

If $\quad y = \dfrac{f(x)}{g(x)}$

then $\dfrac{dy}{dx} = \dfrac{g(x)f'(x) - g'(x)f(x)}{(g(x))^2}$ $\quad$ (a)

alternatively, if $y = \dfrac{u}{v}$ $\quad$ $u \equiv f(x)$, $v \equiv g(x)$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} \quad \text{(b)}$$

The u, v form of Rule IX is again the more favoured form of the rule and is remembered as

$$\mathrm{d}\left(\frac{u}{v}\right) = \frac{v\mathrm{d}u - u\mathrm{d}v}{v^2}$$

or in words, 'dee u upon v equals v dee u minus u dee v all over v squared'.
Rule IX in either form is known as the quotient rule.

Example 7.9
Use the quotient rule to differentiate

(i) $\dfrac{x}{x^2+1}$ (ii) $\dfrac{x^3-1}{x^3+1}$ (iii) $\dfrac{\ln(x^3+4)}{3x+1}$

(i) $u = x,\quad v = x^2+1,$

$\dfrac{\mathrm{d}u}{\mathrm{d}x} = 1,\quad \dfrac{\mathrm{d}v}{\mathrm{d}x} = 2x.$

f(x) = x, g(x) = x^2+1
f'(x) = 1, g'(x) = 2x

The result is

$$\frac{(x^2+1).1 - x.2x}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}.$$

Warning : common errors are
(a) getting the terms on the top in reverse order or
(b) writing the top as $v\dfrac{\mathrm{d}u}{\mathrm{d}x} + u\dfrac{\mathrm{d}v}{\mathrm{d}x}$

(ii) $u = x^3 - 1,\quad v = x^3+1,$

$\dfrac{\mathrm{d}u}{\mathrm{d}x} = 3x^2,\quad \dfrac{\mathrm{d}v}{\mathrm{d}x} = 3x^2.$

The result is

$$\frac{(x^3+1).3x^2 - (x^3-1).3x^2}{(x^3+1)^2} = \frac{3x^5+3x^2-3x^5+3x^2}{(x^3+1)^2} = \frac{6x^2}{(x^3+1)^2}.$$

Use brackets and note the change of sign when the brackets are removed.

(iii) $u = \ln(x^3+4),\quad v = 3x+1,$

$\dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{3x^2}{x^3+4},\quad \dfrac{\mathrm{d}v}{\mathrm{d}x} = 3.$

The result is

$$\frac{(3x+1).\dfrac{3x^2}{x^3+4} - \ln(x^3+4).3}{(3x+1)^2},$$

no significant simplification being possible.

We defer the proof of Rule IX until the following exercises have been worked.

Exercises 7.10

Use the quotient rule to differentiate the following :-

(i) $\dfrac{x-1}{x+1}$ (ii) $\dfrac{x}{\ln x}$ (iii) $\dfrac{e^x}{x+2}$ (iv) $\dfrac{5-3x}{5+3x}$ (v) $\dfrac{1}{x+1}$

(vi) $\dfrac{e^x-1}{e^x+1}$ (vii) $\dfrac{\ln x}{e^x}$ (viii) $\dfrac{x^2-2x+1}{x^2+3}$ (ix) $\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$

The adage 'practice makes perfect' is an apt description of differentiation. For that reason some additional miscellaneous exercises are given here. The various rules are brought together to assist the reader.

	y	$\dfrac{dy}{dx}$
I'	$f(g(x))$	$f'(g(x)) \times g'(x)$
II	e^x	e^x
III	$e^{g(x)}$	$e^{g(x)} \times g'(x)$
IV	$e^{\ln x} = x$	1
	$\ln(e^x) = x$	1
V	$\ln x$	$\dfrac{1}{x}$
VI	$\ln(g(x))$	$\dfrac{g'(x)}{g(x)}$
VII	$f(x) + g(x) + h(x)$	$f'(x) + g'(x) + h'(x)$
VIII	uv / $f(x)g(x)$	$v\dfrac{du}{dx} + u\dfrac{dv}{dx}$ / $g(x)f'(x) + f(x)g'(x)$
IX	$\dfrac{u}{v}$ or $\dfrac{f(x)}{g(x)}$	$\dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$ / $\dfrac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$
	a^x $(a > 0)$	$a^x \ln a$

The differentiation of a^x is the subject of question 8 in the next exercise.

Exercises 7.11

1. Differentiate the following with respect to the appropriate variable.

(i) $x^3 - 3x^2 + x + 2 + \dfrac{1}{x}$ (ii) $x \ln x$ (iii) $\dfrac{x^4-1}{x^4+1}$

(iv) $(x^2+1)^{15}$ (v) $\sqrt{1-x}$ (vi) e^{-4x}

(vii) $x(\ln x)^2$ (viii) $\dfrac{1}{\sqrt{x+1}}$ (ix) $(e^x + 1)\ln(e^x + 1)$

(x) $\dfrac{1}{e^x + e^{-x}}$ (xi) $x(1 - x)$ (xii) $x(1 - x)^{10}$

(xiii) $x + \dfrac{1}{x}$ (xiv) $\left(x + \dfrac{1}{x}\right)\ln x$. (xv) $\dfrac{\ln x}{x^2 + 1}$

2. Find the slope of the tangent to the curve given by $y = \dfrac{2x}{x+1}$ at the point (1, 1). (The slope of the tangent is the value of $\dfrac{dy}{dx}$ at the point in question, see **P1**).

3. Show that if $y = \dfrac{2x+1}{x+1}$ then $\dfrac{d^2y}{dx^2} = -\dfrac{2}{(x+1)^3}$.

4. Find the slope of the tangent to the curve given by $y = e^{x^2+x}$ at the point $(1, e^2)$.

5. Find the maximum and minimum values of the function f given by

$$f(x) = \frac{x}{x^2+1}.$$

6. Find the maximum and minimum values of the function f given by

$$f(x) = x + \frac{1}{x-2} \quad (x \neq 2).$$

7. Find the coordinates of the maximum and minimum points of the curve given by $y = x^2e^{-x}$.

8. From Rule IV, it may be seen that $e^{\ln a} = a$ $(a > 0)$ and thus $a^x = e^{(\ln a)x}$. Use these results to show that

$$\frac{d}{dx}(a^x) = a^x \ln a \quad (a > 0).$$

9. Differentiate the following with respect to x.

(i) 2^x (ii) $x3^x$ (iii) $\dfrac{5^x}{x}$ (iv) $3^x \ln(3x + 1)$ (v) $3^x e^x$.

Postscript to Section 7.4 (non-examinable)

For completeness, we give the proof of the quotient rule (Rule IX) here, i.e. we show that if $y = \dfrac{u}{v}$ then $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$.

$u \equiv f(x)$
$v \equiv g(x)$

We prove the rule by means of the product rule as follows.

Given $\quad y = \dfrac{u}{v},$

we obtain $\quad yv = u.$

Differentiating both sides with respect to x.

$$\therefore \quad \frac{\mathrm{d}}{\mathrm{d}x}(yv) = \frac{\mathrm{d}u}{\mathrm{d}x}.$$

Now yv is a product and the product rule may be used to differentiate it as follows: keep the second (v) fixed and differentiate the first (y), then keep the first fixed and differentiate the second, and add the results.

$$\therefore \quad v\frac{\mathrm{d}y}{\mathrm{d}x} + y\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}.$$

$$\therefore \quad v\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} - y\frac{\mathrm{d}v}{\mathrm{d}x}$$

so $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{v}.\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{y}{v}.\frac{\mathrm{d}v}{\mathrm{d}x} \quad \text{(on dividing through by } v\text{)}.$$

Thus $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{v}.\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{u}{v^2}.\frac{\mathrm{d}v}{\mathrm{d}x} \quad \left(\because y = \frac{u}{v}\right)$$

$$= \frac{v\dfrac{\mathrm{d}u}{\mathrm{d}x} - u\dfrac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$

or $$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{v\dfrac{\mathrm{d}u}{\mathrm{d}x} - u\dfrac{\mathrm{d}v}{\mathrm{d}x}}{v^2}.$$

In terms of $f(x)$ and $g(x)$: -

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.$$

Chapter 8

Differentiation of Trigonometric Functions

This chapter is mainly concerned with the differentiation of sin x, cos x, tan x, sec x, cosec x, cot x.

8.1 Differentiation revisited

Differentiation was introduced in **P1**. The derivative or derived function of the function f(x) was defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

or

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}, \text{ where } y = f(x).$$

8.2 Differentiation of trigonometric functions

To differentiate sin x from first principles we would need to find

$$\lim_{\delta x \to 0} \frac{\sin(x+\delta x) - \sin x}{\delta x}$$

or alternatively

$$\lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}.$$

Here, we do not pursue this first principles approach but settle for stating the result.

> If $y = \sin x$,
>
> $$\frac{dy}{dx} = \cos x.$$

> Also, if $y = \cos x$,
>
> $$\frac{dy}{dx} = -\sin x.$$

Note the negative answer.

Differentiation of tan x may be achieved by using the derived functions of sin x and cos x and the quotient rule (**Chapter 7**).

tan x may be differentiated from first principles but is not done so here.

Let $y = \tan x = \dfrac{\sin x}{\cos x} = \dfrac{u}{v}$ (say).

Then $\dfrac{dy}{dx} = \dfrac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x}$

$y = \dfrac{u}{v}$, $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

$= \dfrac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x}$

See earlier results

$= \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x}$

$\cos^2 x + \sin^2 x = 1$, see P1.

$= \dfrac{1}{\cos^2 x}$

$= \sec^2 x$, since $\sec x = \dfrac{1}{\cos x}$.

The derived function of $\tan x$ is $\sec^2 x$.

We now differentiate $\sec x = \dfrac{1}{\cos x}$. You are not expected to have any other knowledge of $\sec x$ at this stage.

To differentiate $\sec x$, let's observe that $\sec x = \dfrac{1}{\cos x} = (\cos x)^{-1}$.

Then $\dfrac{d}{dx}(\sec x) = \dfrac{d}{dx}((\cos x)^{-1})$

$= (-1)(\cos x)^{-1-1}\dfrac{d}{dx}(\cos x)$

The derived function of $(f(x))^n$ is $n(f(x))^{n-1} \times f'(x)$, see Chapter 7.

$= -(\cos x)^{-2}(-\sin x)$

$= \dfrac{\sin x}{(\cos x)^2}$

$= \sec x \tan x$, since $\tan x = \dfrac{\sin x}{\cos x}$.

The derived function of $\sec x$ is $\sec x \tan x$.

Exercise 8.1
Write $\operatorname{cosec} x = (\sin x)^{-1}$
and deduce that the derived function of $\operatorname{cosec} x$ is
$-\operatorname{cosec} x \cot x$, where $\cot x = \dfrac{\cos x}{\sin x}$.

You are not expected to have any other knowledge of $\operatorname{cosec} x$.

The derived function of $\operatorname{cosec} x$ is $-\operatorname{cosec} x \cot x$.

Finally, let's differentiate cot x by noting that

$\cot x = \dfrac{1}{\tan x} = (\tan x)^{-1}$.

You are not expected to have any other knowledge of cot x.

Then

$$\begin{aligned}\frac{d}{dx}(\cot x) &= \frac{d}{dx}((\tan x)^{-1}) \\ &= (-1)(\tan x)^{-1-1}\frac{d}{dx}(\tan x) \\ &\qquad\qquad\qquad\quad\downarrow \\ &= -\ (\tan x)^{-2}\quad (\sec^2 x) \\ &= \frac{-1}{(\tan x)^2}.\sec^2 x \\ &= -\left(\frac{\cos x}{\sin x}\right)^2 \times \frac{1}{\cos^2 x} \\ &= -\frac{1}{\sin^2 x} \\ &= -\operatorname{cosec}^2 x.\end{aligned}$$

$\dfrac{1}{\tan x} = \dfrac{\cos x}{\sin x}$

The derived function of cot x is $-$ $\operatorname{cosec}^2 x$.

We summarise the above results for convenience.

Function (f(x))	Derivative (f '(x))
sin x	cos x
cos x	$-$ sin x
tan x	$\sec^2 x$
sec x	sec x tan x
cosec x	$-$ cosec x cot x
cot x	$-$ $\operatorname{cosec}^2 x$

In relation to the above table you are expected to

a) know the first three results,

b) be able to derive the last three results from the first three.

We may use these results with the rules given in **Chapter 7** to differentiate more complicated functions.

Example 8.1

Let's recall the function of a function (differentiation of a composite function) rule, namely that the derivative of f(g(x)) is f '(g(x)) × g'(x) or if y = f(u) where u = g(x) then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

Use this rule and the results given earlier to differentiate the following.

(i) $\cos^2 x$ (ii) $\sin(\sqrt{x})$ (iii) $\tan(4x^2 + 2x + 1)$.

(i) The function is f(g(x)) where f(x) = x^2, g(x) = cos x.
The derivative is

$$f'(g(x)) \times g'(x)$$
$$\downarrow \quad \downarrow$$
$$= 2\cos x \times (-\sin x)$$
$$= -2\cos x \sin x.$$

f '(x) = 2x
f '(g(x)) = 2 cos x

Alternatively, $y = u^2$ where $u = \cos x$

and
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\downarrow \quad \downarrow$$
$$= 2u \times (-\sin x)$$
$$= 2\cos x \times (-\sin x)$$
$$= -2\cos x \sin x.$$

(ii) The function is f(g(x)) where f(x) = sin x, g(x) = $\sqrt{x}$.
Then the derivative is

$$f'(g(x)) \times g'(x)$$
$$\downarrow \quad \downarrow$$
$$= \cos\sqrt{x}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$$
$$= \frac{\cos(\sqrt{x})}{2\sqrt{x}}.$$

f '(x) = cos x
f '(g(x)) = cos ($\sqrt{x}$)

Alternatively, $y = \sin u$ where $u = \sqrt{x}$.

Then
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= \cos u \times \frac{1}{2}u^{-\frac{1}{2}}$$
$$= \frac{\cos(\sqrt{x})}{2\sqrt{x}}, \text{ as before.}$$

(iii) Let $y = \tan u$ where $u = 4x^2 + 2x + 1$

so that
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\downarrow \quad \downarrow$$
$$= \sec^2 u \times (8x + 2)$$
$$= 2(4x + 1)\sec^2(4x^2 + 2x + 1).$$

Example 8.2

Use the differentiation of a product rule, namely

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

Chapter 7

to differentiate the following.

(i) $y = (x^2 + 3x + 2)\sin x$ (ii) $y = x^3 \tan x$
(iii) $y = \sin x \cos x$ (iv) $y = \cos^2 x$ $(\cos x \times \cos x)$
(v) $e^{2x} \sin x$ (vi) $(x^2 + 2)\sec 2x$
(vii) $(2x + 1)\operatorname{cosec}(x^3 - x)$ (viii) $e^{x^3} \cot 4x$ (ix) $(\ln x) \sin 3x$.

(i) $u = x^2 + 3x + 2, \quad v = \sin x$

so $\dfrac{du}{dx} = 2x + 3, \quad \dfrac{dv}{dx} = \cos x$

then $\dfrac{dy}{dx} = (\sin x)(2x + 3) + (x^2 + 3x + 2)(\cos x)$

$= (2x + 3)\sin x + (x^2 + 3x + 2)\cos x.$

(ii) $u = x^3, \quad v = \tan x$

so $\dfrac{du}{dx} = 3x^2, \quad \dfrac{dv}{dx} = \sec^2 x$

$\therefore \dfrac{dy}{dx} = (\tan x)(3x^2) + (x^3)(\sec^2 x)$

$= x^2(3\tan x + x\sec^2 x).$

(iii) $u = \sin x, \quad v = \cos x$

so $\dfrac{du}{dx} = \cos x, \quad \dfrac{dv}{dx} = -\sin x.$

$\therefore \dfrac{dy}{dx} = (\cos x)(\cos x) + (\sin x)(-\sin x)$

$= \cos^2 x - \sin^2 x.$

(iv) $u = \cos x, \quad v = \cos x$

so that $\dfrac{du}{dx} = -\sin x, \quad \dfrac{dv}{dx} = -\sin x.$

$\therefore \dfrac{dy}{dx} = (\cos x)(-\sin x) + (\cos x)(-\sin x)$

$= -2\cos x\sin x.$

(v) $u = e^{2x}, \quad v = \sin x$

so $\dfrac{du}{dx} = 2e^{2x}, \quad \dfrac{dv}{dx} = \cos x.$

$\therefore \dfrac{d}{dx}(e^{2x}\sin x) = (\sin x)(2e^{2x}) + (e^{2x})(\cos x)$

$= e^{2x}(2\sin x + \cos x).$

(vi) $u = x^2 + 2, \quad v = \sec 2x$

so that $\dfrac{du}{dx} = 2x, \quad \dfrac{dv}{dx} = (\sec 2x\tan 2x)2 = 2\sec 2x\tan 2x.$

$\dfrac{d}{dx}((x^2 + 2)\sec 2x) = (\sec 2x)(2x) + (x^2 + 2)(2\sec 2x\tan 2x)$

$= 2\sec 2x[x + (x^2 + 2)\tan 2x].$

(vii) $u = (2x + 1), \quad v = \operatorname{cosec}(x^3 - x)$

so that $\dfrac{du}{dx} = 2, \quad \dfrac{dv}{dx} = -\operatorname{cosec}(x^3 - x)\cot(x^3 - x) \times (3x^2 - 1)$

$$\frac{d}{dx}((2x+1)\operatorname{cosec}(x^3-x)) = (\operatorname{cosec}(x^3-x)(2)$$
$$+ (2x+1)(-\operatorname{cosec}(x^3-x)\cot(x^3-x)) \times (3x^2-1)$$
$$= \operatorname{cosec}(x^3-x)[2-(2x+1)(3x^2-1)\cot(x^3-x)].$$

(viii) $u = e^{x^3}$, $v = \cot 4x$

so that $\frac{du}{dx} = e^{x^3} \times 3x^2 = 3x^2e^{x^3}$

and $\frac{dv}{dx} = (-\operatorname{cosec}^2 4x) \times 4 = -4\operatorname{cosec}^2 4x.$

Note again the use of the function of a function rule.

$\therefore$ $\frac{d}{dx}\left(e^{x^3}\cot 4x\right) = (\cot 4x)\left(3x^2e^{x^3}\right) + \left(e^{x^3}\right)(-4\operatorname{cosec}^2 4x)$

$= e^{x^3}[3x^2\cot 4x - 4\operatorname{cosec}^2 4x].$

(ix) $u = \ln x$, $v = \sin 3x$

so that $\frac{du}{dx} = \frac{1}{x}$, $\frac{dv}{dx} = 3\cos 3x.$

$\therefore$ $\frac{d}{dx}((\ln x)\sin 3x) = (\sin 3x)\left(\frac{1}{x}\right) + (\ln x)(3\cos 3x)$

$= \frac{1}{x}\sin 3x + 3(\ln x)\cos 3x.$

Example 8.3

Use the differentiation of a quotient rule, namely

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

to differentiate the following.

(i) $\frac{\sin x}{x^2}$ (ii) $\frac{\sin x + \cos x}{x}$ (iii) $\frac{e^{2x}}{\cos 3x}$

(iv) $\frac{\tan x - 1}{\sec x}$ (v) $\frac{e^{-2x}}{\cos 2x + \sin 2x}$

(i) $u = \sin x$, $v = x^2$

so that $\frac{du}{dx} = \cos x$, $\frac{dv}{dx} = 2x.$

$\therefore$ $\frac{d}{dx}\left(\frac{\sin x}{x^2}\right) = \frac{(x^2)\cos x - (\sin x)(2x)}{x^4}$

$= \frac{x\cos x - 2\sin x}{x^3}.$

cancelling one x

(ii) $\quad u = \sin x + \cos x, \quad v = x$

so that $\quad \dfrac{du}{dx} = \cos x - \sin x, \quad \dfrac{dv}{dx} = 1.$

$$\therefore \quad \frac{d}{dx}\left(\frac{\sin x + \cos x}{x}\right) = \frac{(x)(\cos x - \sin x) - (\sin x + \cos x)(1)}{x^2}$$

$$= \frac{(x-1)\cos x - (x+1)\sin x}{x^2}.$$

(iii) $\quad u = e^{2x}, \quad v = \cos 3x$

so that $\quad \dfrac{du}{dx} = 2e^{2x}, \quad \dfrac{dv}{dx} = -3\sin 3x.$

$$\therefore \quad \frac{d}{dx}\left(\frac{e^{2x}}{\cos 3x}\right) = \frac{(\cos 3x)(2e^{2x}) - (e^{2x})(-3\sin 3x)}{\cos^2 3x}$$

$$= \frac{e^{2x}(2\cos 3x + 3\sin 3x)}{\cos^2 3x}.$$

(iv) $\quad u = \tan x - 1, \quad v = \sec x$

so that $\quad \dfrac{du}{dx} = \sec^2 x, \quad \dfrac{dv}{dx} = \sec x \tan x.$

$$\text{Then} \quad \frac{d}{dx}\left(\frac{\tan x - 1}{\sec x}\right) = \frac{(\sec x)(\sec^2 x) - (\tan x - 1)(\sec x \tan x)}{\sec^2 x}$$

$$= \frac{\sec x(\sec^2 x - \tan^2 x + \tan x)}{\sec^2 x}$$

$$= \frac{\sec^3 x - \sec x \tan^2 x + \sec x \tan x}{\sec^2 x}.$$

(v) $\quad u = e^{-2x}, \quad v = \cos 2x + \sin 2x$

so that $\quad \dfrac{du}{dx} = -2e^{-2x}, \quad \dfrac{dv}{dx} = -2\sin 2x + 2\cos 2x.$

$$\therefore \quad \frac{d}{dx}\left(\frac{e^{-2x}}{\cos 2x + \sin 2x}\right)$$

$$= \frac{(\cos 2x + \sin 2x)(-2e^{-2x}) - (e^{-2x})(-2\sin 2x + 2\cos 2x)}{(\cos 2x + \sin 2x)^2}$$

$$= \frac{e^{-2x}(-2\cos 2x - 2\sin 2x + 2\sin 2x - 2\cos 2x)}{(\cos 2x + \sin 2x)^2}$$

$$= \frac{-4e^{-2x}\cos 2x}{(\cos 2x + \sin 2x)^2}.$$

Exercises 8.2

Differentiate the following.

1. (i) $3\sin x$ (ii) $\cos 3x$ (iii) $\sin\left(\frac{x}{2}\right)$ (iv) $\tan\left(\frac{x}{4}\right)$

(v) $\sec\left(\frac{3}{4}x\right)$ (vi) $\operatorname{cosec} 2x$ (vii) $\sin 3x + \cos 3x$ (viii) $\sec x + \tan x$

(ix) $2\cos\frac{x}{2}$ (x) $x^{\frac{1}{3}}\cos x$ (xi) $2x^2\cos x + (x^2+1)\sin x$

(xii) $\frac{2}{x^3} + 5x\sin x - x\tan x$ (xiii) $\cos^3(x^2)$ (xiv) $\sqrt{\tan x}$

(xv) $x^2\cos 2x$ (xvi) $x\sqrt{\sin x}$ (xvii) $\sin^2 x + \cos^2 x$

(xviii) $\frac{\cos x}{2x+3}$ (xix) $\frac{e^{3x}}{\cos x}$ (xx) $\frac{\cos 2x}{\sqrt{x}}$

(xxi) $\frac{\sin x}{\sqrt{\cos x}}$ (xxii) $\ln(\cos x)$ (xxiii) $\ln(\cot x + \operatorname{cosec} x)$

(xxiv) $\sqrt{2+\sin^2 x}$.

8.3 Maxima and minima problems involving trigonometric functions

We consider further application of the techniques introduced in **P1** to investigate stationary values of functions.

Let's first recall the approaches to be adopted in investigating stationary values of a function f(x).

Method 1	Method 2
(a) $f'(x) = 0$, then	(a) $f'(x) = 0$, then
either (b) $f'(x)$ changes from + to –, maximum	either (b) $f''(x) < 0$ for a maximum
or (c) $f'(x)$ changes from – to +, minimum	or (c) $f''(x) > 0$ for a minimum
or (d) $f'(x)$ doesn't change sign, stationary point of inflexion.	or (d) $f''(x) = 0$, no information.

The second method uses second derivatives to classify stationary values. Second derivatives of trigonometric functions are easily found, in principle at least.

Example 8.4

(i) If $y = e^x(\cos x + \sin x)$, find $\frac{d^2y}{dx^2}$.

(ii) Given $f(x) = \frac{1-\sin x}{1+\sin x}$, find $f''(x)$.

(i) $$\frac{dy}{dx} = (\cos x + \sin x)(e^x) + (e^x)(-\sin x + \cos x)$$

$u = e^x$, $v = \cos x + \sin x$

$\frac{du}{dx} = e^x$, $\frac{dv}{dx} = -\sin x + \cos x$

$$= 2e^x\cos x.$$

Similarly, $\frac{d^2y}{dx^2} = (2\cos x)(e^x) + (e^x)(-2\sin x)$

$= 2e^x(\cos x - \sin x)$.

(ii) $f'(x) = \dfrac{(1+\sin x)(-\cos x)-(1-\sin x)(\cos x)}{(1+\sin x)^2}$

$u = 1-\sin x$, $v = 1+\sin x$, $\dfrac{du}{dx} = -\cos x$, $\dfrac{dv}{dx} = \cos x$

$= \dfrac{-\cos x - \sin x\cos x - \cos x + \sin x\cos x}{(1+\sin x)^2}$

so $f'(x) = \dfrac{-2\cos x}{(1+\sin x)^2}$.

function of function rule with u^2 where $u = 1+\sin x$

For this function let $u = -2\cos x$, $v = (1+\sin x)^2$

so that $\dfrac{du}{dx} = 2\sin x$, $\dfrac{dv}{dx} = 2(1+\sin x)(\cos x)$.

Then $f''(x) = \dfrac{(1+\sin x)^2(2\sin x)-(-2\cos x)2(1+\sin x)(\cos x)}{(1+\sin x)^4}$

$= \dfrac{(1+\sin x)}{(1+\sin x)^4}[2\sin x(1+\sin x)+4\cos^2 x]$

cancelling $1+\sin x$

$= \dfrac{2\sin x + 2\sin^2 x + 4\cos^2 x}{(1+\sin x)^3}$

$= \dfrac{2(\sin x + \sin^2 x + 2\cos^2 x)}{(1+\sin x)^3}$

$\sin^2 x + \cos^2 x = 1$

$= \dfrac{2(1+\sin x + \cos^2 x)}{(1+\sin x)^3}$.

We use the second derivative test (method 2) to investigate the stationary values of a function.

Example 8.5

Find the stationary values or turning points on the curve of

$y = \sin x + \cos x \quad \text{for} \quad 0 \le x \le 2\pi$.

For turning points, $\dfrac{dy}{dx} = 0$.

Now $\dfrac{dy}{dx} = \cos x - \sin x$

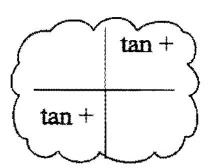
and $\dfrac{d^2y}{dx^2} = -\sin x - \cos x$.

Then $\dfrac{dy}{dx} = 0$

gives $\cos x - \sin x = 0$

$\therefore \cos x = \sin x$

and $\dfrac{\sin x}{\cos x} = \tan x = 1$.



Then $x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$ in the range 0 to 2π.

When $x = \frac{\pi}{4}$,

$$\frac{d^2y}{dx^2} = -\sin\frac{\pi}{4} - \cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}} = -\sqrt{2} < 0,$$

corresponding to a maximum point.
The corresponding maximum value is

$$y = \cos\frac{\pi}{4} + \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}.$$

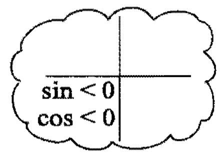
Don't approximate the surd if not asked to do so.

When $x = \frac{5\pi}{4}$,

$$\frac{d^2y}{dx^2} = -\sin\frac{5\pi}{4} - \cos\frac{5\pi}{4}$$

$$= -\left(-\frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}}\right)$$

$$= \sqrt{2} > 0, \text{ corresponding to a minimum point.}$$



$$y = \cos\frac{5\pi}{4} + \sin\frac{5\pi}{4}$$

$$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}.$$

The stationary or turning points are therefore

$\left(\frac{\pi}{4}, \sqrt{2}\right)$ maximum point,

$\left(\frac{5\pi}{4}, -\sqrt{2}\right)$ minimum point.

In the next example, we use the sign test on the first derivative to classify the stationary values.

Example 8.6
Find the stationary values of

$$f(x) = \tan^2 x - 2\tan x \qquad \text{for } 0 \le x \le \frac{\pi}{2}.$$

For a stationary value,

$$f'(x) = 0.$$

Now
$$f'(x) = (2\tan x)(\sec^2 x) - 2\sec^2 x$$
$$= 2\sec^2 x(\tan x - 1).$$

To differentiate $\tan^2 x$ we could write u^2 where $u = \tan x$.

Then $f'(x) = 0$ gives,

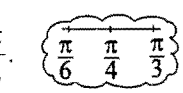
$$2\sec^2 x(\tan x - 1) = 0.$$

$\therefore$ $\sec^2 x = \frac{1}{\cos^2 x} = 0$ (impossible)

or $\tan x = 1$.

$\therefore$ $x = \frac{\pi}{4}$.

To use the sign test on $f'(x)$, let's consider the signs of $f'(x)$ at $x = \frac{\pi}{6}, \frac{\pi}{3}$.



$\underline{x = \frac{\pi}{6}}$

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2},\quad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

or use your calculator

$\sec x = \frac{1}{\cos x}$

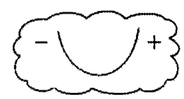
so that
$$f'(x) = 2 \times \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2}\left(\frac{1}{\sqrt{3}} - 1\right)$$
$$= \frac{8}{3}\left(\frac{1}{\sqrt{3}} - 1\right) < 0.$$

$\underline{x = \frac{\pi}{3}}$

$$\cos\frac{\pi}{3} = \frac{1}{2},\quad \tan\frac{\pi}{3} = \sqrt{3}$$

so that
$$f'(x) = 2 \times \frac{1}{\left(\frac{1}{2}\right)^2}\left(\sqrt{3} - 1\right)$$
$$= 8\left(\sqrt{3} - 1\right) > 0.$$

Thus as x passes through $\frac{\pi}{4}$, f $'(x)$ changes sign from − to +, corresponding to a minimum value.



The corresponding stationary value is
$$f\left(\frac{\pi}{4}\right) = \tan^2\frac{\pi}{4} - 2\tan\frac{\pi}{4} = 1 - 2 = -1.$$

Thus f(x) has a minimum value of − 1 when $x = \frac{\pi}{4}$.

Alternatively,

second derivative test

$$f'(x) = \underbrace{2\tan x}_{u}\ \underbrace{\sec^2 x}_{v} - 2\sec^2 x$$

product rule

$$f''(x) = \sec^2 x\frac{d}{dx}(2\tan x) + 2\tan x\frac{d}{dx}(\sec^2 x) - 2\frac{d}{dx}\left(\sec^2 x\right)$$

function of a function rule

$$= 2\sec^4 x + (2\tan x)(2\sec x \times \sec x\tan x) - 2(2\sec x \times \sec x \tan x)$$

giving $f''(x) = 2\sec^4 x + 4\sec^2 x\tan^2 x - 4\sec^2 x\tan x.$

When $x = \frac{\pi}{4}$, $\sec x = \frac{1}{\cos x} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}.$

$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$
$\tan\frac{\pi}{4} = 1$

$$f''(x) = 2(\sqrt{2})^4 + 4(\sqrt{2})^2(1)^2 - 4(\sqrt{2})^2(1)$$
$$= 8 + 8 - 8 = 8 > 0,$$

corresponding to a minimum point as before.

Exercises 8.3

1. Find the maximum and minimum values of the following functions for $0 \le x \le 2\pi$.
(i) $\cos 2x - x$ (ii) $\sqrt{3}\sin x + \cos x$ (iii) $e^x(2\cos x + \sin x)$
(iv) $\cos x + \sin x \cos x$. (v) $\cos^2 x + 4\cos x + 6$.

2. Prove that the function $8 \sec\theta + 27 \operatorname{cosec}\theta$ has stationary values when $\tan\theta = \frac{3}{2}$. If θ is acute, calculate the stationary value.

3. Show that the least value of $3\sec x - 2\tan x$ for $0 < x < \frac{\pi}{2}$ is approximately 2.24.

4. The turning effect of a ship's rudder is shown theoretically to be $k\cos\theta\sin^2\theta$, where θ is the angle which the rudder makes with the keel, and k is a constant. For what value of θ is the rudder most effective?
Note that the values of θ of interest lie in the range $-90°$ to $90°$.

5. Find the maximum and minimum values of $\cos^3 x + \sin^3 x$ for $0 \le x \le \frac{\pi}{2}$.

Chapter 9

More Integration

Integration was introduced in **P1.** There, the process of integration was confined to integration of constants and polynomial functions. Here, we extend the list of basic functions to be integrated: $\frac{1}{x}$, e^x, $\sin x$, $\cos x$ are considered.

Secondly, a new rule of integration is introduced whereby certain types of composite functions may be integrated. Later, some definite integrals are evaluated. The Trapezium Rule which enables the calculation of approximate values of definite integrals is considered.

9.1 Techniques and rules

Let's start by recalling a rule from **P1.**

Rule I

$$\int cx^n \mathrm{d}x = \frac{cx^{n+1}}{n+1} + k, (n \neq -1)$$

where c and k are constants.

The reader may wish to perform a quick revision of this rule by working the following exercises. Before doing so, we recall that when differentiating a finite number of terms, we may differentiate term by term. Since integration reverses differentiation, we may integrate term by term.

Exercises 9.1

Integrate the following with respect to the appropriate letter, ignoring the constants of integration.

(i) x^6 (ii) $x^{1/3}$ (iii) $\frac{4}{x^4}$ (iv) $\sqrt{x}$

(v) $x + \frac{1}{x^2}$ (vi) $\left(y^2 + \frac{1}{y^2}\right)^2$

It is possible to generalise Rule I to functions of the form $(ax+b)^n$ where a, b and n are constants. We consider an example before giving the general result.

Example 9.1

Now $\frac{\mathrm{d}}{\mathrm{d}x}((7x+5)^4) = 4(7x+5)^3.7$

$= 28(7x+5)^3$

or $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{(7x+5)^4}{28}\right) = (7x+5)^3.$

Thus writing in terms of integrals, we have

$$\int (7x+5)^3 dx = \frac{(7x+5)^4}{28} + k.$$

Note also that a constant in the integral would not introduce any difficulty. Thus

$$\int 3(7x+5)^3 dx = \frac{3(7x+5)^4}{28} + k.$$

The general rule is therefore

Rule II $$\int c(ax+b)^n dx = \frac{c(ax+b)^{n+1}}{(n+1)a} + k$$

where a, b and c are constants.

BEWARE

It is stressed that Rule II holds because $ax + b$ is a so-called linear function in x, i.e. x occurs only as x^1. This eliminates $\frac{3}{x} + 2$, for example, because $\frac{3}{x}$ is $3x^{-1}$.

Note that $$\int (ax^2+b)^n dx \neq \frac{(ax^2+b)^{n+1}}{(n+1)2ax} + k$$

because $$\frac{d}{dx}\frac{(ax^2+b)^{n+1}}{(n+1)2ax} \neq (ax^2+b)^n.$$

You are urged strongly not to use such a fallacious result or any similar result which does not relate to linear functions.

Exercises 9.2

1. Integrate the following by using Rules I and II. Note that there is no need to remove the brackets before integrating. Ignore the constants of integration.

(i) $(x + 1)^2$ (ii) $(2x - 1)^3$ (iii) $(3x + 7)^4$

(iv) $(7x - 6)^{-6}$ (v) $(3x+1)^{\frac{1}{2}}$ (vi) $(9x-8)^{-\frac{1}{2}}$

(vii) $\frac{1}{(2x+3)^{\frac{3}{2}}}$ (viii) $\frac{1}{\sqrt{1+x}}$ (ix) $(3-2x)^{\frac{3}{2}} + (3-2x)^{\frac{1}{2}}$

(x) $(lx + m)^s$ (l, m, s are constants, $s \neq -1$).

2. Which of the following may be evaluated by means of rule II?

(i) $\int (2x+3)^{\frac{1}{2}} dx$ (ii) $\int (5x^3-1)^{\frac{1}{2}} dx$ (iii) $\int (7-x)^4 dx$

(iv) $\int (3+x^2)^{\frac{1}{2}} dx$.

9.2 Integration of $\frac{1}{x}$, e^x, $\sin x$ and $\cos x$

It was pointed out earlier in **P1** that Rule I : $\int cx^n \,\mathrm{d}x = \dfrac{cx^n}{n+1} + k$ is valid only if $n \neq -1$, since division by zero is undefined. Thus Rule I does not assist with finding $\int \frac{1}{x}\,dx$.

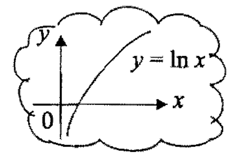
In **Chapter 7**, it was shown that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \frac{1}{x}$$

$\int x^n \,\mathrm{d}x$ **with** $n = -1$

so that

$$\int \frac{1}{x}dx = \ln x + k.$$



A difficulty arises that ln x is undefined for $x < 0$. In mathematics we must consider the integral even when $x < 0$. To cope with this requirement we write

$$\int \frac{1}{x}dx = \int \frac{-1}{-x}dx = \int \frac{d(-x)}{-x}$$
$$= \ln(-x) + k.$$

Thus

$$\int \frac{1}{x}dx = \ln x + k \qquad x > 0$$
$$= \ln(-x) + k. \qquad x < 0$$

These statements may be combined into the single statement

Rule III $\boxed{\int \frac{1}{x}dx = \ln|x| + k.}$

$|x| = x,\ x \geq 0$
$= -x,\ x < 0$
See Chapter 1.

Rule III may be developed to consider integrals such as $\dfrac{dx}{3x+2}$ and $\int \dfrac{dx}{5-7x}$

These integrals are special cases of the more general, $\int \dfrac{dx}{ax+b}$ where a and b are constants. Note that $ax + b$ is a linear function.

$$\text{Now}\quad \frac{\mathrm{d}}{\mathrm{d}x}\ln(ax+b) = \frac{1}{ax+b} \times \frac{\mathrm{d}}{\mathrm{d}x}(ax+b)$$
$$= \frac{a}{ax+b}.$$

Rule V1
Chapter 7

Thus

$$\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + k$$

and more generally,

Rule IV $\boxed{\int \frac{c}{ax+b}\mathrm{d}x = \frac{c}{a}\ln|ax+b| + k}$

We urge you again <u>not</u> to use such a result with anything other than linear functions.

$\int \dfrac{c\,\mathrm{d}x}{ax^2+b} \neq \dfrac{c}{2ax}\ln|ax^2+b| + k$

Example 9.2

$$\int \frac{5}{3x+2}\,dx \quad = \frac{5}{3}\ln|3x+2| + k.$$

$$\int \frac{6}{5-7x}\,dx \quad = -\frac{6}{7}\ln|5-7x| + k.$$

Exercises 9.3

Integrate the following.

(i) $\frac{2}{x}$ (ii) $\frac{1}{3x}$ (iii) $\frac{1}{x+1}$ (iv) $\frac{1}{9x+7}$

(v) $\frac{1}{1-x}$ (vi) $\frac{1}{3-x} + \frac{1}{3+2x}$

Now we consider the function e^x. Since

$$\frac{d}{dx}(e^x) = e^x$$

we have

Rule V $\quad \boxed{\int e^x dx = e^x + k.}$

Now since $\quad \frac{d}{dx}(ce^{ax+b}) = ce^{ax+b}a$

$$= cae^{ax+b}$$

Rule III Chapter 7

then

Rule VI $\quad \boxed{\int ce^{ax+b} dx = \frac{c}{a}e^{ax+b} + k}$

We ask you to note again our earlier comments concerning linear functions : do not use Rule VI for non-linear functions.

We note in passing that the integral of a^x $(a > 0)$ will not be considered in the following exercises.

Exercises 9.4

1. Use Rule VI to integrate the following.

(i) e^{2x+1} (ii) e^{-x+3} (iii) e^{-2x} (iv) e^{5-3x}

(v) $\frac{1}{e^{4x}}$ (vi) $e^{3x} + \frac{1}{e^{3x}}$ (vii) $(e^x)^{\frac{5}{2}}$ (viii) $2e^{5x} - 6e^{-2x}$

2. Which of the following may be evaluated using Rule VI?

(i) e^{2x-1} (ii) e^{-10x} (iii) $e^{9x} - 6e^{-4x}$

(iv) $e^{\frac{1}{x}+1}$ (v) $e^{\frac{1}{x^2}}$ (vi) e^{71-3x}

It was pointed out in **Chapter 8** that

$$\frac{d}{dx}(\sin x) = \cos x$$

so that

Rule VII $\boxed{\int \cos x dx = -\sin x + k.}$

Also $\frac{d}{dx}\cos x = -\sin x$

so that

Rule VIII $\boxed{\int \sin x dx = -\cos x + k.}$

Note the negative sign in the result

Then since $\frac{d}{dx}(\sin(ax+b)) = a\cos(ax+b)$

$\frac{d}{dx}(\cos(ax+b)) = -a\sin(ax+b)$

Rule I' Chapter 7

we have the following generalisations of Rules VII, VIII:

Rule IX $\boxed{\int \cos(ax+b)\, dx = \frac{1}{a}\sin(ax+b) + k.}$

Rule X $\boxed{\int \sin(ax+b) dx = -\frac{1}{a}\cos(ax+b) + k.}$

Exercises 9.5

Integrate the following by Rules IX and X, omitting the constants of integration.

1. (i) $\sin(x+2)$ (ii) $\cos 5x$ (iii) $\sin(9-5x)$
(iv) $\cos(4x-7) - 3\sin(2x+5)$ (v) $2\cos(7x+1) + 5\sin 3x$

2. Which of the following may be evaluated using Rules IX and X?
(i) $\sin(3x+1)$ (ii) $\cos(9-5x^2)$ (iii) $\sin(2x^2+1)$
(iv) $\sin\left(\frac{1}{x}\right)$ (v) $\cos\left(\frac{1}{x^2}\right)$ (vi) $\cos(9-7x)$

As in differentiation, 'practice makes perfect'. For this reason some additional exercises are given below. For completeness, the rules established in this Chapter are first summarised. The arbitrary constants are omitted to avoid repetition.

	function f(x)	∫f(x) dx
Rule I	cx^n	$\frac{cx^{n+1}}{n+1}\quad (n \neq -1)$
Rule II	$c(ax+b)^n$	$\frac{c(ax+b)^{n+1}}{(n+1)a}\quad (n \neq -1)$
Rule III	$\frac{1}{x}$	$\ln\|x\|$
Rule IV	$\int \frac{c}{ax+b}dx$	$\frac{c}{a}\ln\|ax+b\|$
Rule V	$\int e^x dx$	e^x
Rule VI	$\int ce^{ax+b}dx$	$\frac{c}{a}e^{ax+b}$
Rule VII	$\int \cos x dx$	$\sin x$
Rule VIII	$\int \sin x dx$	$-\cos x$
Rule IX	$\int \cos(ax+b)dx$	$\frac{1}{a}\sin(ax+b)$
Rule X	$\int \sin(ax+b)dx$	$-\frac{1}{a}\cos(ax+b)$

Exercises 9.6

Integrate the following functions with respect to the appropriate letter.

(i) $\frac{1}{x^2}$ (ii) $\frac{1}{x}$ (iii) $x^{\frac{3}{4}}$ (iv) $x+3$

(v) $\sqrt{x+3}$ (vi) $x^2 - \frac{3x}{2}$ (vii) $\frac{1}{10x-9}$ (viii) e^{x+3}

(ix) e^{5-9x} (x) $(3x+2)^{10}$ (xi) $\frac{1}{(2x+9)^3}$

(xii) $\frac{2x^3+6x^2+3}{x^3}$ (xiii) $\sqrt{x}\left(2x+x^{-\frac{3}{2}}\right)$ (xiv) $\left(x+\frac{1}{x}\right)^3$

(xv) $\frac{1}{\sqrt[3]{x}}$ (xvi) $(a+bt)^2$ (xvii) $\frac{1}{\sqrt{3-2y}}$ (xviii) $y(2-5y^2)$

(xix) $\frac{3}{2+3y}$ (xx) $\frac{x^2-4}{x^4}$ (xxi) $\frac{1}{t\sqrt{t}}+\frac{5}{\sqrt{3+5t}}$ (xxii) $\frac{5}{(13-5w)^3}$

(xxiii) $\sin 5x$ (xxiv) $3\sin\left(2y-\frac{\pi}{4}\right)$

(xxv) $4\cos 3y - 6\sin(7y+5)$ (xxvi) $7\sin(3-2x)+2\cos(10-x)$

9.3 Definite Integrals and the Trapezium Rule

In this section we return to the topic of definite integration introduced in **P1.**

We recall that the definite integral $\int_a^b f x \mathrm{d}x$ may be interpreted as the area under the curve $y = f(x)$ between $x = a$ and $x = b$.

Example 9.3

Evaluate (i) $\int_2^3 \frac{1}{2x+3}\mathrm{d}x$ (ii) $\int_0^{\frac{\pi}{4}} \sin 2x \mathrm{d}x$ (iii) $\int_1^2 e^{5-3x}\mathrm{d}x$

(i)
$$\int_2^3 \frac{1}{2x+3}\mathrm{d}x = \left[\frac{1}{2}\ln(|2x+3|)\right]_2^3$$
$$= \frac{1}{2}\ln 9 - \frac{1}{2}\ln 7$$
$$\simeq 0.126, \text{ using the calculator.}$$

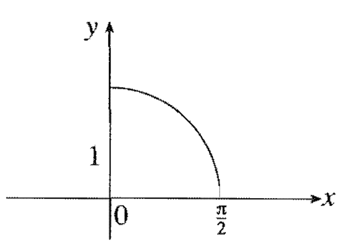
(ii)
$$\int_0^{\frac{\pi}{4}} \sin 2x \mathrm{d}x = \left[-\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{4}}$$
$$= -\frac{1}{2}\cos\frac{\pi}{2} - \left(-\frac{1}{2}\cos 0\right)$$
$$= \frac{1}{2}.$$

(iii)
$$\int_1^2 e^{5-3x}\mathrm{d}x = \left[-\frac{1}{3}e^{5-3x}\right]_1^2$$
$$= -\frac{1}{3}e^{-1} - \left(-\frac{1}{3}e^2\right)$$
$$= \frac{1}{3}e^2 - \frac{1}{3}e^{-1} \simeq 2.340, \text{ using the calculator.}$$

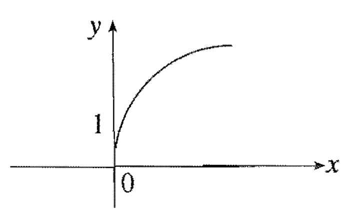
Example 9.4

Sketch the curve $y = \cos x - \sin x$ for $0 \le x \le \frac{\pi}{2}$, indicating clearly where the curve crosses the x-axis. Hence evaluate the area enclosed by the curve, the y-axis and $x = 0$, $x = \frac{\pi}{2}$.

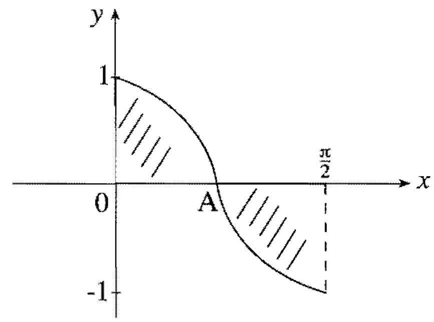
The curve $y = \cos x - \sin x$ is a combination of the $y = \cos x$ and $y = \sin x$ curves.



$y = \cos x$



$y = \sin x$



The required area is shown shaded. The x coordinate of the point A where the curve crosses the x-axis is given by

$$\cos x - \sin x = 0.$$

$$\therefore \quad \cos x = \sin x.$$

$$\therefore \quad \frac{\sin x}{\cos x} = \tan x = 1.$$

$$\therefore \quad x = \frac{\pi}{4}.$$

The total shaded area is

$$\int_0^{\frac{\pi}{4}} (\cos x - \sin x)\, dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cos x - \sin x)\, dx$$

$$= \quad [\sin x + \cos x]_0^{\frac{\pi}{4}} - [\sin x + \cos x]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \quad \sin\frac{\pi}{4} + \cos\frac{\pi}{4} - \sin 0 - \cos 0 - \sin\frac{\pi}{2} - \cos\frac{\pi}{2} + \sin\frac{\pi}{4} + \cos\frac{\pi}{4}$$

$$= \quad 2\sin\frac{\pi}{4} + 2\cos\frac{\pi}{4} - 2$$

$$= \quad \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} - 2 = 2\sqrt{2} - 2$$

$$\simeq 0.828$$

$\cos 0 = 1$, $\sin\frac{\pi}{2} = 1$, $\cos\frac{\pi}{2} = 0$, $\sin 0 = 0$

$\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$.

Exercises 9.7

1. Evaluate $\int_0^3 \frac{1}{(x+4)^2}\, dx$.

2. Find the area enclosed between the curve $y = \sin x$ and the x-axis bounded by the lines $x = 0$ and $x = \pi$.

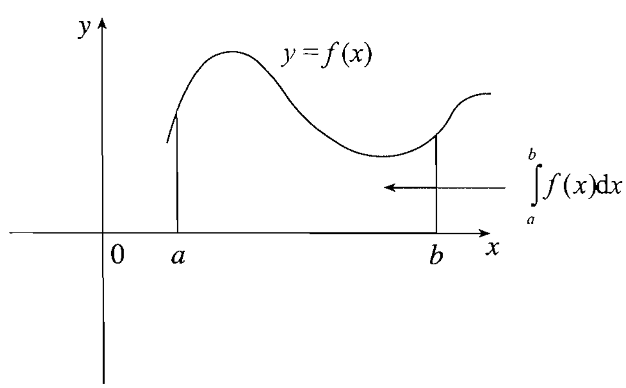
3. Evaluate $\int_0^2 \frac{(e^x + e^{-x})^2}{e^x}\, dx$.

4. Evaluate the integrals (a) $\int_0^1 \frac{9}{2+3x}\mathrm{d}x$ (b) $\int_0^1 \frac{9}{\sqrt{2+3x}}\mathrm{d}x$.

5. Sketch the curve $y = e^x$. Find the area between the curve and the lines $x = 1$ and $y = 1$.

Sometimes we are unable to evaluate definite integrals by first finding the associated indefinite integral; for instance, it is not possible to find the indefinite integral $\int \sqrt{\sin x}\,\mathrm{d}x$. If you think you've found it, differentiate your answer and see whether you obtain $\sqrt{\sin x}$!

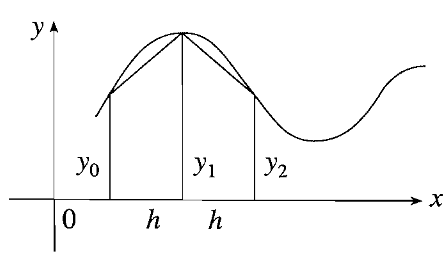
To deal with such cases, we recall that a definite integral $\int_a^b f(x)\mathrm{d}x$ is a number which represents the area between $y = f(x)$ and $x = a$, $x = b$. (It is assumed here that $y = f(x)$ does not cross the x-axis.)



Thus when finding an approximate value of the area, we are finding an approximate value of $\int_a^b f(x)\mathrm{d}x$.

The approximate value of the area may be found by various methods. Here, we consider the <u>Trapezium Rule</u> for finding an approximation.

The approximation is found by joining the ends of consecutive coordinates and treating each trapezium formed as an approximation for the area under the corresponding part of the curve.



If the first three ordinates are y_o, y_1, y_2, and h is the distance between consecutive ordinates, the areas of the first two trapezia are $\frac{h}{2}(y_o + y_1)$, $\frac{h}{2}(y_1 + y_2)$. Supposing there are n trapezia or strips, and therefore n+1 ordinates, we see that the total area of the trapezia is

$$\frac{h}{2}[(y_0 + y_1) + (y_1 + y_2) + \ldots\ldots + (y_{n-1} + y_n)],$$

where y_n is the last ordinate.

Then the area under the curve = $\int_a^b f(x)dx$ is given approximately by

$$\frac{h}{2}(y_0 + 2y_1 + 2y_2 + \ldots.. + 2y_{n-1} + y_n).$$

y_0 ↓ first ordinate; y_n ↓ last ordinate

This formula is known as the **trapezium rule.**

Example 9.5
Use the trapezium rule with five ordinates to find an approximate value for

$$\int_{\frac{\pi}{2}}^{\pi} \sqrt{\sin x}\,dx.$$

We note that there are five ordinates or four strips. Then $h = \dfrac{\frac{\pi}{2} - \frac{\pi}{4}}{4} = \dfrac{\pi}{16}$.

The approximate value is $\frac{1}{2}.\frac{\pi}{16}[y_0 + 2y_1 + 2y_2 + 2y_3 + y_4]$ where the y values are the values of $\sqrt{\sin x}$ at $x = \frac{\pi}{2}, \frac{5\pi}{8}, \frac{6\pi}{8}, \frac{7\pi}{8}, \pi$.

x	$\frac{\pi}{2}$	$\frac{5\pi}{8}$	$\frac{6\pi}{8}$	$\frac{7\pi}{8}$	π
$y = \sqrt{\sin x}$	1	0.9611865	0.8408964	0.6186141	0
Factor	1	2	2	2	1

Then $\int_{\frac{\pi}{2}}^{\pi} \sqrt{\sin x}\,dx \simeq \frac{1}{2} \times \frac{\pi}{16}[1 + 2(0.9611865 + 0.8408964 + 0.6186141) + 0]$

$= 0.57348\ldots$

$\simeq 0.5735$, rounding to four decimal places.

Exercises 9.8

1. Use the trapezium rule with seven strips to find an approximate value for

$$\int_0^{0.7} \ln(1+x)\mathrm{d}x\,.$$

2. Use the trapezium rule with five ordinates to find an approximate value for

$$\int_0^{0.8} e^{x^2}\,\mathrm{d}x\,.$$

3. Use the trapezium rule with eleven ordinates to find an approximate value for

$$\int_0^{\frac{1}{2}} \frac{\mathrm{d}x}{\sqrt{1-x^2}}\,.$$

Given that the value of the integral is $\frac{\pi}{6}$, find an approximate value for π.

4. Use the trapezium rule with six ordinates to find an approximate value for

$$\int_1^{2} \sqrt{1+x^3}\,\mathrm{d}x\,.$$

5. A function $y = f(x)$ is tabulated for various values of x as shown below.

x	1.0	1.2	1.4	1.6	1.8
y	3.70	3.82	4.15	4.51	5.07

Estimate $\int_{1.0}^{1.8} y\mathrm{d}x$, using the trapezium rule.

Chapter 10

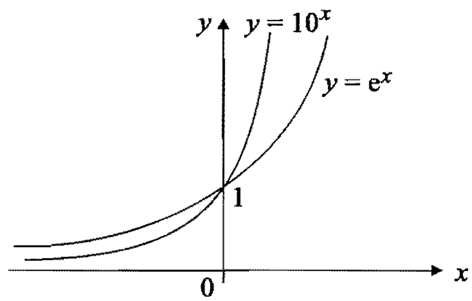
Laws of Logarithms

The exponential and logarithmic (log) functions were considered in **Chapters 5 and 7**. There, the graphs and differentiation of the functions were discussed. Here we develop some laws satisfied by logs.
We start by recalling some of the ideas arising earlier.

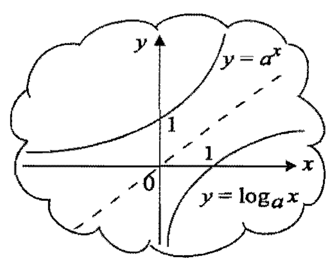
10.1 Exponential and logarithmic functions : recap

The most general form of the exponential function is $f(x) = a^x$, where a is a positive constant. The cases in which $a = e$ and $a = 10$ are of particular interest, the first because of the differentiation of e^x, the second because 10 is the basis of our everyday number system.

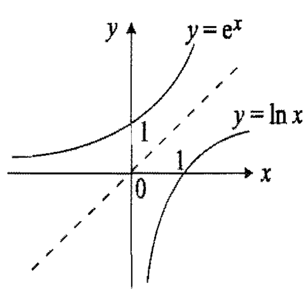
The graphs of $y = e^x$ and $y = 10^x$ are similar : they fall to the left, climb to the right, and pass through the point (1, 0). In fact, all graphs of the form $y = a^x$ ($a > 0$) pass through the point (0, 1).

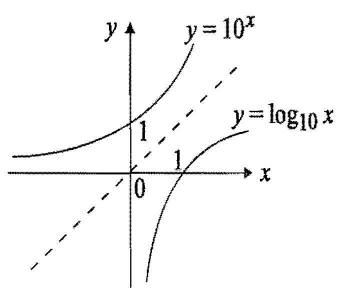


The function $f(x) = a^x$ ($a \neq 1$) is a one-one function and has an inverse function $\log_a x$, known as the log function. The log is said to be to the **base** a in this case.



The inverses of $f(x) = e^x$ and $g(x) = 10^x$ together with their inverses are shown below.





The following features of the graphs are important.

$\ln 1 = 0, \; \log_{10} 1 = 0.$ $\ln x \to \infty, \; \log_{10} x \to \infty$ as $x \to \infty.$ $\ln x \to -\infty, \; \log_{10} x \to -\infty$ as $x \to 0.$ $\ln x$ and $\log_{10} x$ are not defined for $x < 0.$	(I)

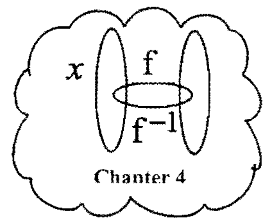
In fact $\log_a 1 = 0$ for any base a.

The usual relations between functions and their inverses are particularly important for exponential and logarithmic functions.

Thus when $f(x) = e^x$, $f^{-1}(x) = \ln x$

and the usual results $f f^{-1}(x) = x$,

$f^{-1} f(x) = x$,



lead to and

$$e^{\ln x} = x, \quad \ln(e^x) = x \qquad * \quad \text{(II)}$$

* This important result will be used in Section 10.2.

with similar results for 10^x and $\log_{10} x$, and indeed for any base.

$a^{\log_a x} = x$, $\log_a(a^x) = x$

Putting $x = 1$ in (II) above, we obtain the following important results :-

$$e^{\ln 1} = 1, \quad \ln(e^1) = 1. \qquad \text{(III)}$$

These results state in effect that $e^0 = 1$ and also that $\log_e e = 1$.

$\ln 1 = 0$

More generally for $a > 0$,

$$a^{\log_a 1} = 1, \quad \log_a a = 1. \qquad \text{(IV)}$$

$10^{\log_{10} 1} = 1$, $\log_{10} 10 = 1$.

Exercises 10.1

1. Simplify the following without use of a calculator.

(i) $\log_{10}(10^{1.314})$ (ii) $\ln(e^{4.92})$ (iii) $\ln(e^{-5.61})$ (iv) $e^{\ln 3.61}$

(v) $10^{\log_{10}(5.16)}$ (vi) $15^{\log_{15} 1}$ (vii) $\log_{30} 30$

2. Write the following in logarithmic notation.

(i) $a = e^x$ (ii) $b = 10^y$ (iii) $c = d^z$ (iv) $10^0 = 1$

(v) $e^2 = 7.389056$ (correct to six decimal places)

(vi) $10^{2.31} = 204.1738$ (correct to four decimal places)

The derivatives of $f(x) = e^x$ and $f^{-1}(x) = \ln x$ were discussed in **P1** (Chapter 7).

Then, given

$$y = e^x, \quad \frac{dy}{dx} = e^x;$$

$$y = e^{f(x)}, \quad \frac{dy}{dx} = e^{f(x)}\, f'(x);$$

function of function rule

$$y = \ln x, \quad \frac{dy}{dx} = \frac{1}{x};$$

$$y = \ln g(x), \quad \frac{dy}{dx} = \frac{g'(x)}{g(x)}.$$

function of function rule

Exercises 10.2

1. Differentiate the following with respect to x.
   (i) $\ln(x+1)$ (ii) $\ln(x^2+1)$ (iii) $\ln(3x)$ (iv) $\ln(4x^2)$

2. By first writing $10 = e^{\ln 10}$, express 10^x as a power of e and hence show that
$$\frac{d}{dx}(10^x) = 10^x \ln 10.$$

3. Show that the following pairs of functions have the same derivatives. You may assume $x > 0$.
   (i) $\ln(6x)$, $\ln x$
   (ii) $\ln\left(\frac{x}{7}\right)$, $\ln x$
   (iii) $\ln(Ax)$, $\ln x$
   (iv) $\ln\left(\frac{x}{B}\right)$, $\ln x$
   (v) $\ln(x^2)$, $2 \ln x$
   (vi) $\ln(x^n)$, $n \ln x$
   (vii) $\ln(Ax^2)$, $2 \ln x$

A, *B* are positive constants

Before moving to section 10.2, let's consider question 3, exercises 10.2 in a little more detail. Specifically, let's look at question 3(i).

The functions $\ln(6x)$ and $\ln x$ have the same derivative $\left(\frac{1}{x}\right)$ but are different functions. Our knowledge of differentiation tells us that they must differ by a constant which disappears on differentiation.

$\therefore \quad \ln(6x) = \ln x + \text{constant}.$

x^2 and $x^2 + 3$ have the same derivative, for example.

Similarly for other parts of the question :-

$\ln\left(\frac{x}{7}\right) = \ln x + \text{constant},$

$\ln(x^2) = 2 \ln x + \text{constant},$

$\ln(Ax^2) = 2\ln x + \text{constant}.$

The above relationships are explored in the next section where the laws of logs are established.

10.2 Laws of logs

It turns out that the laws of logs are a consequence of the laws of indices considered in **P1.** Those laws are

(addition of indices) $b^m \times b^n = b^{m+n}$,

(subtraction of indices) $b^m \div b^n = b^{m-n}$,

(multiplication of indices) $(b^m)^n = b^{mn}$.

(i) Log of a product of two numbers

Consider the product pq. From the properties of the exponential and logarithmic functions,

$$pq = e^{\ln pq}, \quad (1)$$
$$p = e^{\ln p}, \quad (2)$$
$$q = e^{\ln q}. \quad (3)$$

See Rule II in section 10.1 with $x = pq$, $x = p$, $x = q$ in turn.

Now $pq = e^{\ln p} \times e^{\ln q}$ (using (2), (3))

$= e^{\ln p + \ln q}$ (addition of indices)

so $pq = e^{\ln p + \ln q}$ (4)

Comparison of (4) and (1) gives

$$\ln(pq) = \ln p + \ln q \quad \text{(V)}$$
i.e. the log of a product = sum of logs.

Similarly $\log_{10}(pq) = \log_{10} p + \log_{10} q$

(ii) Log of a quotient

Consider the quotient $\frac{p}{q}$. Now, as before,

$$\frac{p}{q} = e^{\ln \frac{p}{q}}, \quad (5)$$
$$p = e^{\ln p}, \quad (6)$$
$$q = e^{\ln q}. \quad (7)$$

Then $\frac{p}{q} = \frac{e^{\ln p}}{e^{\ln q}}$ (from (6), (7))

$= e^{\ln p - \ln q}$ (subtraction of indices)

so $\frac{p}{q} = e^{\ln p - \ln q}$. (8)

Comparison of (8) and (5) gives

$$\ln\left(\frac{p}{q}\right) = \ln p - \ln q \qquad \text{(VI)}$$

or the log of a quotient = log(numerator) − log(denominator).

Similarly $\log_{10}\left(\frac{p}{q}\right) = \log_{10} p - \log_{10} q$

(iii) Log of a power

We consider p^n.

Now, as before, $p = e^{\ln p}$ (9)

and $p^n = e^{\ln(p^n)}$. (10)

From (9), $p^n = (e^{\ln p})^n$

$= e^{n \ln p}$,

multiplication of indices

so $p^n = e^{n \ln p}$. (11)

From (10), $p^n = e^{\ln(p^n)}$. (10)

Comparison of (11) and (10) gives

$$\ln(p^n) = n \ln p \qquad \text{(VII)}$$

i.e. log(of a power of a given number) = power × log(of the given number).

Similarly $\log_{10} p^n = n \log_{10} p$

Example 10.1

Let's return to question 3, exercises 10.2.

(i) $\ln(6x) = \ln 6 + \ln x$ (Rule V)

(ii) $\ln\left(\frac{x}{7}\right) = \ln x - \ln 7$ (Rule VI)

(iii) $\ln(Ax) = \ln A + \ln x$ (Rule V)

(iv) $\ln\left(\frac{x}{B}\right) = \ln x - \ln B$ (Rule VI)

(v) $\ln(x^2) = 2 \ln x$ (Rule VII)

(vi) $\ln(x^n) = n \ln x$ (Rule VII)

(vii) $\ln(Ax^2) = \ln A + 2 \ln x$ (Rule V followed by Rule VII)

The constants in (i)–(iv), (vii) will disappear on differentiation leading to the results quoted in question 3, exercises 10.2.

The arguments for the derivation of the Rules V – VII are easily extended to situations involving more than two numbers. The following example illustrates the case when more than two numbers are involved.

Example 10.2

$$\ln\left(\frac{3^2 5^3 11^{-6}}{7^3 13^7}\right) = 2\ln 3 + 3\ln 5 - 6\ln 11 - 3\ln 7 - 7\ln 13.$$

Example 10.3

1. Express the following in terms of $\ln x$, $\ln y$, $\ln z$ or $\log_{10} x$, $\log_{10} y$ and $\log_{10} z$.

(i) $\ln xy$ (ii) $\ln xyz$ (iii) $\ln \frac{x}{y}$ (iv) $\ln \frac{xy}{z}$ (v) $\ln \frac{x}{yz}$ (vi) $\ln \frac{1}{x}$

(vii) $\ln \frac{x^2}{y}$ (viii) $\ln \frac{x^3}{\mathrm{e}}$ (ix) $\ln \frac{x}{\mathrm{e}y}$ (x) $\log_{10} \frac{10^2}{\sqrt{x}}$ (xi) $\log_{10} \frac{x^2 y^3}{10\sqrt{z}}$

Answers

(i) $\ln x + \ln y$ (ii) $\ln x + \ln y + \ln z$ (iii) $\ln x - \ln y$

(iv) $\ln x + \ln y - \ln z$ (v) $\ln x - \ln y - \ln z$ (vi) $-\ln x$

(vii) $2\ln x - \ln y$ (viii) $3\ln x - 1$ $(\ln \mathrm{e} = 1)$

(ix) $\ln x - 1 - \ln y$ (x) $2 - \frac{1}{2}\log_{10} x$ $(\log_{10} 10^2 = 2)$

(xi) $2\log_{10} x + 3\log_{10} y - 1 - \frac{1}{2}\log_{10} z$

Logs of products and/or quotients of functions may often be differentiated without too much difficulty if the logs are first expanded.

Example 10.4

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ if $y = \ln\left(\sqrt{\frac{(x+1)(x+2)}{(2x-3)}}\right)$ $\left(x > \frac{3}{2}\right)$.

Now $y = \ln\left(\sqrt{\frac{(x+1)(x+2)}{(2x-3)}}\right)$

$= \frac{1}{2}\ln(x+1) + \frac{1}{2}\ln(x+2) - \frac{1}{2}\ln(2x-3)$. (Rules V, VI, VII)

$\therefore \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2(x+1)} + \frac{1}{2(x+2)} - \frac{1}{2x-3}.$

differentiation of each log in turn

Exercises 10.3

1. Write the following in terms of $\ln a$, $\ln b$, $\ln x$, $\ln y$, $\ln z$, $\log_{10} x$, $\log_{10} y$, $\log_{10} z$ where appropriate.

(i) $\ln \frac{1}{x^4}$ (ii) $\ln xy^{\frac{3}{2}}$ (iii) $\ln x^4 y^{\frac{3}{2}}$ (iv) $\ln \sqrt[3]{x}$ (v) $\ln \frac{x^{\frac{2}{3}} y^4}{z^3}$

(vi) $\ln(\mathrm{e}a)$ (vii) $\ln \frac{1}{\mathrm{e}^2 b^2}$ (viii) $\log_{10} \sqrt{\frac{x}{y}}$ (ix) $\log_{10} x^2 \sqrt{\frac{y^3}{2}}$

(x) $\log_{10}\left(\frac{x\sqrt{y}}{\sqrt[3]{z}}\right)$

2. Choose the correct options in the following.

Options

(i) $\ln 2 + \ln 3$ — (a) $\ln 5$ (b) $\ln\left(\frac{3}{2}\right)$ (c) $\ln 6$ (d) $\ln\left(\frac{2}{3}\right)$

(ii) $\ln 18 - \ln 9$ — (a) $\ln 9$ (b) $\ln 2$ (c) $2\ln 2$ (d) $\frac{\ln 18}{\ln 19}$

(iii) $\ln\left(\frac{27}{5}\right)$ — (a) $\frac{\ln 27}{\ln 5}$ (b) $\ln 22$ (c) $\ln 27 - \ln 5$ (d) $\left(\frac{5}{27}\right)$

(iv) $\ln x + \ln y - \ln z$ — (a) $\ln xyz$ (b) $\ln(x+y-z)$ (c) $\ln\frac{z}{xy}$ (d) $\ln\frac{xy}{z}$

(v) $\frac{1}{2}\ln x - \frac{1}{3}\ln y + \ln e^2$ — (a) $\ln\frac{2\sqrt{x}}{\sqrt[3]{y}}$ (b) $\ln\sqrt{x} - \ln\sqrt[3]{y} + 2$ (c) $\ln\frac{2x}{\sqrt[3]{y}}$ (d) $\ln\frac{e^2\sqrt{x}}{\sqrt{y}}$

(vi) $2 + 3\log_{10} x$ — (a) $\log_{10}\frac{2}{3}x$ (b) $\log_{10} 6x$ (c) $\log_{10}\frac{x^3}{2}$ (d) $\log_{10} 100x^3$

3. By first simplifying the following by means of rules V – VI, find the values of x.

(i) $\ln(x+1) + \ln(x-1) = \ln 2$ (ii) $\ln(x+1) - \ln(x^2-1) = 1$

(iii) $\ln x^3 = 2 + \ln x$

4 Given $\ln x + 2\ln y - \ln 5 = 0$, express y in terms of x in the form $y = Ax^n$, giving the values of A and n.

5. Express $\ln y = 2\ln(x+1) - 3\ln x + \ln(x^2+1) + 2$ in the form $y = f(x)$.

6. By first expanding the given log in each case, differentiate the following. You need not simplify your answers.

(i) $\ln 6x^2$ (ii) $\ln(x+1)(x+2)$ (iii) $\ln\left(\frac{(2x+1)(x+2)^4}{(3x-5)}\right)$

(iv) $\ln((x-2)^3(2x+5)^2)$ (v) $\ln\frac{(x+1)^3(3x-2)^4}{(2x+1)^2}$

7. Show that if $y = \ln\left(\sqrt{\frac{x+1}{x-1}}\right)$, $\frac{dy}{dx} = \frac{-1}{x^2-1}$.

Chapter 11

Solution of Equations

Linear equations such as

$$3x + 5 = 7 - x$$

and quadratic equations such as

$$x^2 + 5x + 3 = 0$$

are simple examples of polynomial equations. Methods of solving such equations were considered in **P1.**

This chapter introduces methods of solving equations involving polynomials of higher degree. One of the methods introduced may also be used to solve equations not involving polynomials, for example equations such as

$$e^{-x} = \ln x.$$

It is also shown that the laws of logarithms may be used to solve certain types of equation.

11.1 Polynomial equations : use of the factor theorem

Quadratic equations may be solved by factorising or by means of the quadratic formula. No such formula is available for the solution of equations involving higher degree polynomials. However, on occasions, the polynomial may factorise.

Example 11.1

Solve the equation

$$2x^3 + x^2 - 15x - 18 = 0.$$

This is an equation involving a third degree polynomial.

Such an equation is called a cubic equation.

Let $\quad f(x) = 2x^3 + x^2 - 15x - 18.$

We factorise the expression by means of the factor theorem (see **Chapter 2**).

$$\begin{aligned} \text{Now} \quad f(1) &= 2 + 1 - 15 - 18 = -30 \neq 0. \\ f(-1) &= -2 + 1 + 15 - 18 = -4 \neq 0. \\ f(2) &= 16 + 4 - 30 - 18 = -28 \neq 0. \\ f(-2) &= -16 + 4 + 30 - 18 = 0. \end{aligned}$$

Then since $f(-2) = 0$, $x + 2$ is a factor of the polynomial.

We divide out the factor to obtain

$$\begin{aligned} 2x^3 + x^2 - 15x - 18 &= (x + 2)(2x^2 - 3x - 9) \\ &= (x + 2)(x - 3)(2x + 3), \end{aligned}$$

on factorising the quadratic expression. **P1**

Long division or matching terms, Chapter 2.

The equation becomes

$$(x+2)(x-3)(2x+3) = 0$$

so that $x = -2$ or 3 or $-\frac{3}{2}$.

If $abc = 0$, then $a = 0$, or $b = 0$, or $c = 0$.

The roots are therefore $-2, 3, -\frac{3}{2}$.

Example 11.2

Solve the equation

$$x^4 + 6x^3 + 7x^2 - 8x - 6 = 0.$$

Let $f(x) = x^4 + 6x^3 + 7x^2 - 8x - 6$.

Now $f(1) = 1 + 6 + 7 - 8 - 6 = 0$

so $(x - 1)$ is a factor of the polynomial.

Dividing out the factor, we obtain

$$x^4 + 6x^3 + 7x^2 - 8x - 6 = (x-1)(x^3 + 7x^2 + 14x + 6).$$

Check this.

The factor theorem is then applied to

$$g(x) = x^3 + 7x^2 + 14x + 6.$$

A little work along the previous lines shows that

$$g(-3) = (-3)^3 + 7(-3)^2 + 14(-3) + 6$$
$$= -27 + 63 - 42 + 6 = 0.$$

When attempting to factorise $g(x)$ you should check the possibility that $x - 1$ is also a factor of $g(x)$.

Thus $(x + 3)$ is a factor of $x^3 + 7x^2 + 14x + 6$.

Then $x^4 + 6x^3 + 7x^2 - 8x - 6 = (x-1)(x+3)(x^2 + 4x + 2)$,

after dividing out the factors.

The expression $x^2 + 4x + 2$ does not contain any obvious factors.

Then $x - 1 = 0$, i.e. $x = 1$

or $x + 3 = 0$, i.e. $x = -3$

or $x^2 + 4x + 2 = 0$,

giving $x = \frac{-4 \pm \sqrt{16-8}}{2} = \frac{-4 \pm \sqrt{8}}{2} = -2 \pm \sqrt{2}$.

Do not approximate the surds unless you are asked to do so.

The roots are therefore $1, -3$ and $-2-\sqrt{2}, -2+\sqrt{2}$.

Exercises 11.1

1. Solve the equation $x^3 - 3x^2 - 4x + 12 = 0$.

2. Solve the equation $x^3 - 2x^2 + 1 = 0$.

3. Find the value of k if $x = -2$ is a root of $x^3 + kx^2 + 6x - 4 = 0$.

4. Solve the equation $2x^3 + 3x^2 - 32x + 15 = 0$.

5. Given that $x = \pm 1$ are roots of the equation
$$x^3 + ax^2 + bx - 2 = 0,$$
find the other root of the equation.

6. Show that the equation $x^4 - 4x^3 + x^2 + 16x - 20 = 0$ has only two real roots.

When equations cannot be solved exactly, we find approximate values of their roots. The roots are then found to any required degree of accuracy. Some methods of finding approximate values of roots involve the use of an initial approximation. A method of finding an initial approximation is considered in the next section.

11.2 Location of the roots of f(x) = 0

As an example, we consider the equation

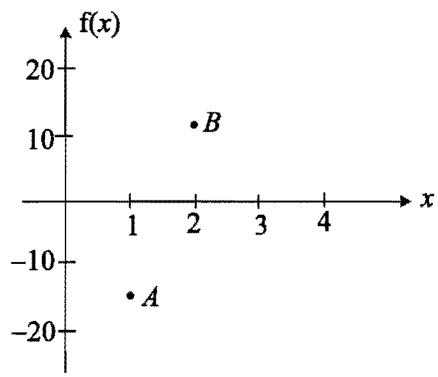
$$f(x) = 0,$$

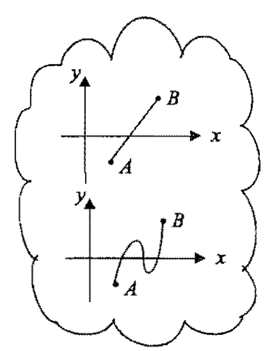
where $f(x) = 4x^3 + 6x^2 - 20x - 5.$

Now $f(1) = 4(1)^3 + 6(1)^2 - 20(1) - 5 = -15$

and $f(2) = 4(2)^3 + 6(2)^2 - 20(2) - 5 = 11$

so that f(1) and f(2) are of opposite sign.





Thus for a graph of $y = 4x^3 + 6x^2 - 20x - 5$, the point A is below the x-axis whilst the point B lies above the x-axis. Since the graph of $y = 4x^3 + 6x^2 - 20x - 5$ contains no breaks, ie. is a continuous line, it must cross the x-axis at least once between $x = 1$ and $x = 2$. In other words, there is a value for x between $x = 1$ and $x = 2$ at which $4x^3 + 6x^2 - 20x - 5 = 0$.

This example is a special case of a more general situation.

If f is any continuous function (so the graph of f is a continuous line) **and f(a), f(b) are of opposite sign, there is at least one root of f(x) = 0 in (a, b).**

This statement provides a method for the location of roots of equations of the type $f(x) = 0$, where f is a continuous function.

Example 11.3

By finding the value of $f(x) = x^4 - 3x^3 + x^2 + x - 3$ at $x = 0, 1, 2, \ldots$ show that $f(x) = 0$ has at least one positive root. Find a root correct to one decimal place.

x	f(x)
0	−3
1	−3
2	−5
Root here → - - - -	- - - -
3	9

The actual values of f(x) are unimportant: the signs of the values are the main interest, but find them.

As f(2) and f(3) are of opposite sign and f is a continuous function, there is a root of $x^4 - 3x^3 + x^2 + x - 3 = 0$ between $x = 2$ and $x = 3$. To find the root correct to 1 decimal place using this approach, we could consider the values of f(2.1), f(2.2), ..., f(2.9) and seek a change of sign between adjacent values of f(x).

x	f(x)
2.1	−4.8
2.2	−4.5
2.3	−3.9
2.4	−3.1
2.5	−2.1
2.6	−0.7
Root here → - - - -	- - - -
2.7	1.1

N.B.
A convenient method of calculation of values of $x^4 - 3x^3 + x^2 + x - 3$ is to write the polynomial in the nested form $(((x - 3)x + 1)x + 1)x - 3$.

Since there is a change of sign of f(x) between $x = 2.6$ and $x = 2.7$, the root is between 2.6 and 2.7. It appears that the root is closer to 2.6 than 2.7 because f(2.6) = −0.7 is nearer to 0 than f(2.7) = 1.1.
To check this we find the value of f at the point halfway between $x = 2.6$ and $x = 2.7$, i.e. at $x = 2.65$. In fact f(2.65) = 0.2. Since there is a change of sign of f(x) between $x = 2.6$ and $x = 2.65$, the root lies between 2.6 and 2.65. The root is nearer 2.6 than 2.7 and is therefore 2.6, to one decimal place.

x	**f(x)**
2.6	**−0.7**
2.65	**0.2**

Alternatively, the number of calculations could have been reduced by noting that there was a change of sign of f(x) between $x = 2$ and $x = 3$, and finding the value of f(x) at the midpoint of [2, 3], i.e. at 2.5.

x	f(x)
2	−5
2.5	−2.1
2.6	−0.7
Root here → - - - -	- - - -
2.7	1.1

As there is no change of sign of f(x) between $x = 2$ and $x = 2.5$, we calculate f(2.6), f(2.7),...

As before, $x = 2.6$ is an approximate root and we calculate f(2.65) as before. Then $x = 2.6$ is an approximate root to one decimal place.
This method of location of roots, by noting changes of sign of the function values, becomes tedious when greater accuracy is required. Fortunately, other methods of solving equations exist. The change of sign method given here is useful in that it provides a first approximation for use with the more refined methods.

Exercises 11.2

1. Given that $x = 2$ is an approximate root of the equation

$$x^3 - 2x^2 + x - 1 = 0,$$

find the value of this root correct to one decimal place.

2. By making a table of values of $f(x) = x^3 - 4x^2 + x + 1$ for integer values of x between -2 and 5 inclusive, locate the three roots of $x^3 - 4x^2 + x + 1 = 0$. Find the largest root correct to one decimal place.

3. Find a root, correct to one decimal place, between -3 and 0 of the equation

$$4x^3 + 6x^2 - 20x - 5 = 0.$$

4. By first considering $f(x) = x^2 - 2$, show that $\sqrt{2}$ lies between 1.41 and 1.42.

11.3 Iterative methods

Up until now, equations have been solved by direct methods such as factorising, quadratic formula and graphical methods. The method of iteration adopts a different approach : an initial approximation to the root is refined until an answer to the required accuracy is obtained.

The method used here depends upon first rewriting an equation

$$f(x) = 0$$

in the form

$$x = g(x).$$

Example 11.4

Given the equation

$$x^3 - 9x^2 + 24x - 13 = 0$$

has a root between 0 and 1, find this root correct to 3 decimal places.

We note in passing that if

$$f(x) = x^3 - 9x^2 + 24x - 13$$

then $f(0) = -13$, $f(1) = 3$.

Since $f(x)$ is continuous and $f(0)$, $f(1)$ differ in sign, there is indeed a root between 0 and 1.

Section 11.2

Let's get back to finding this solution.

Now the equation

$$x^3 - 9x^2 + 24x - 13 = 0$$

may be rewritten as

$$24x = -x^3 + 9x^2 + 13$$

so

$$x = \frac{1}{24}(-x^3 + 9x^2 + 13). \qquad (1)$$

$f(x) = 0$ has been rewritten as $x = g(x)$.

We add the suffices n and $n + 1$ to the x terms in (1) to obtain

$$x_{n+1} = \frac{1}{24}\left(-x_n^3 + 9x_n^2 + 13\right). \qquad (2)$$

$n + 1$ on the left hand side, n on the right hand side.

Then if $n = 0$ in (2),

$$x_1 = \frac{1}{24}\left(-x_0^3 + 9x_0^2 + 13\right) \qquad (3)$$

so that if we substitute a value for x_0, we can find a value for x_1.

What shall we use for x_0? Well, there is a root between 0 and 1 so let's take

$$x_0 = \frac{0+1}{2} = 0.5.$$

the mean of the end values.

Substitution in (3) for x_0 gives

$$x_1 = \frac{1}{24}(-(0.5)^3 + 9(0.5)^2 + 13)$$
$$= 0.63021.$$

We shall carry 5 decimal places in our working

When $n = 1$ in (2),

$$x_2 = \frac{1}{24}\left(-x_1^3 + 9x_1^2 + 13\right) = 0.68017,$$

on substituting $x_1 = 0.63021$.

Similarly, we substitute for x_2 to find x_3 from (2) with $n = 2$, i.e. from

$$x_3 = \frac{1}{24}\left(-x_2^3 + 9x_2^2 + 13\right).$$

This process may be continued along the above lines and the work set out as follows.

n	x_n	x_{n+1}
0	0.5	0.63021
1	0.63021	0.68017
2	0.68017	0.70204
3	0.70204	0.71207
4	0.71207	0.71676
5	0.71676	0.71898
6	0.71898	0.72003
7	0.72003	0.72053
8	0.72053	0.72077
9	0.72077	0.72088
10	0.72088	0.72093
11	0.72093	0.72096
12	0.72096	0.72097

$x_{n+1} = \frac{1}{24}(-x_n^3 + 9x_n^2 + 13)$

The output at any stage is the input for the next stage.

In practice, you need not use a table such as this because the results at each stage may be stored in the calculator.

The process is terminated when the outputs (x_{n+1} values) are in agreement to a specified number of decimal places. In the table, it is clear that after $n = 7$ the changes between the x_{n+1} values are not affecting the third decimal place. The root appears to be $x = 0.721$, correct to 3 decimal places.
We check that the root is 0.721, correct to three decimal places, as follows.
The root appears to be between 0.720 and 0.721.
Let's find the values of

$$f(x) = x^3 - 9x^2 + 24x - 13$$

when $x = 0.7205$ and 0.721, since the root **appears** to be nearer 0.721 than 0.720.

0.7209 . . .

The equation is $x^3 - 9x^2 + 24x - 13 = 0$.

x	f(x)
0.7205	− 0.006
← - - - - - - - - - -	← root
0.721	+ 0.0002

The root lies between 0.7205 and 0.721, i.e. is nearer 0.721 than 0.720, and is therefore 0.721, correct to three decimal places.

One of the strengths of this **iteration** method is that it may be used with equations which do not involve polynomials.

Example 11.5

It is known that the equation

$$x - \sin x - 0.2 = 0,$$

where x is measured in radians, has an approximate root equal to 1.1. Find this root correct to 4 decimal places.

The dictionary definition of iteration is 'repeat'.

Rewrite the equation in the form

$$x = \sin x + 0.2$$

to set up the iterative process

$$x_{n+1} = \sin x_n + 0.2.$$

Then using $x_0 = 1.1$, we have

We quote results to 6 decimal places.

$$x_1 = \sin(1.1) + 0.2 = 1.091207,$$

$$x_2 = \sin(1.091207) + 0.2 = 1.087184.$$

Similarly, $x_3 = 1.085321$, $\quad x_4 = 1.084453$,

$x_5 = 1.084048$, $\quad x_6 = 1.083859$,

$x_7 = 1.083770$, $\quad x_8 = 1.083728$,

$x_9 = 1.083709$, $\quad x_{10} = 1.083700$.

Note that x_8, x_9, x_{10} agree to within 4 decimal places and we believe that $x = 1.0837$, correct to 4 decimal places.

Let's check our belief.

The root appears to be between 1.0837 and 1.0838. We find the values of

$$f(x) = x - \sin x - 0.2$$

when $\quad x = 1.0837$ and 1.08375.

x	f(x)	
1.0837	+ 0.0000043 . .	
1.08375	+ 0.00003907	← **no root here, try 1.08365**
1.08365	− 0.000022 . .	

There is a root between 1.08365 and 1.0837.

Thus the root is 1.0837 correct to four decimal places.

You may feel that the checking procedures used in examples 11.4 and 11.5 are unnecessary. However, there is a tendency to terminate an iterative process prematurely before the correct root is found. The checking procedure should eliminate any possible error of that type.

There is no unique way of rewriting an equation for purposes of iteration.

Example 11.6

In this example alternative rearrangements of the equations considered in examples 11.4 and 11.5 are given.

(a) Show that $x_{n+1} = \frac{1}{2}(x_n + \sin x_n + 0.2)$ is a possible iterative process for solving the equation $x - \sin x - 0.2 = 0$.

(b) Show that $x_{n+1} = \dfrac{9x_n^2 + 13}{x_n^2 + 24}$ is a possible iterative process for solving the equation

$$x^3 - 9x^2 + 24x - 13 = 0.$$

Given an iterative formula we are able to find the underlying equation by dropping the suffices.

(a) The underlying equation is

$$x = \frac{1}{2}(x + \sin x + 0.2)$$

so $\quad 2x = x + \sin x + 0.2$

$\therefore \quad x - \sin x - 0.2 = 0.$

(b) The underlying equation is

$$x = \frac{9x^2 + 13}{x^2 + 24}$$

so that $\quad x^3 + 24x = 9x^2 + 13$

$\therefore \quad x^3 - 9x^2 + 24x - 13 = 0.$

Given there are many ways of rewriting an equation to obtain an iterative process, which one should be used? The answer to that question is outside the scope of this course. Suffice it to say, not all rearrangements are useful!

There are an infinite number in fact.

Example 11.7

The equation

$x^3 - 9x^2 + 24x - 13 = 0$

has an approximate root 0.7.

See Example 11.4 the root is 0.721, correct to 3 decimal places.

The rearrangement

$$x = \frac{1}{x^2}(9x^2 - 24x + 13)$$

of the equation gives the iteration formula

$$x_{n+1} = \frac{1}{x_n^2}\left(9x_n^2 - 24x_n + 13\right).$$

Using $x_0 = 0.7$ with this formula, we obtain

$x_1 = 1.244898, \quad x_2 = -1.890352$

$x_3 = 25.334009, \quad x_4 = 8.072912.$

It is clear that even though the starting value was taken close to the root, the successive values of $x_1, x_2, \ldots$ bear no relationship to this root.

Thus this particular rearrangement is not useful.

Exercises 11.3

1. Solve the equation $x^2 - 2x - 1 = 0$ by rearranging it in the form
$$x = \frac{1}{x-2}$$
and using $x_0 = -0.4$ to start the iteration. Give your answer correct to three decimal places.

2. Show that $x_{n+1} = \dfrac{2x_n^3 + 1}{3x_n^2 - 1}$ is an iteration formula arising from a rewriting of the equation $x^3 - x - 1 = 0$. Use this iteration formula with starting value $x_0 = 1.3$ to find a root correct to four decimal places.

3. Show that $x_{n+1} = \dfrac{\sin x_n - x\cos x_n + 0.2}{1 - \cos x_n}$
is an iteration formula arising from a rearrangement of the equation
$$x - \sin x - 0.2 = 0.$$
Use this formula, with starting value $x_0 = 1.1$, to find a root of this equation correct to four decimal places.

4. Show that the iteration formula
$$x_n = e^{-x_n}$$
with starting value $x_0 = 0.57$ can be used to find a root of
$$xe^x = 1$$
correct to three decimal places.
Show further that the iteration formula
$$x_{n+1} = -\ln x_n$$
can be derived from the equation but that it cannot be used with $x_0 = 0.57$ to find the root.

5. By first considering the change of sign of
$$f(x) = \tan x - 2x$$
find an approximate value to the positive root of
$$\tan x - 2x = 0 \quad \text{for } 1 < x < 2.$$
Use the iteration formula
$$x_{n+1} = \frac{x\sec^2 x_n + x_n - \tan x_n}{\sec^2 x_n - 1}$$
with $x_0 = 1.2$ to find a root of this equation correct to three decimal places.

6. Sketch the graph of $y = x^3 - 3x^2 - 1$ and deduce that the equation
$$x^3 - 3x^2 - 1 = 0$$
has only one root. Show that this root lies between 3 and 4. Use $x_0 = 3.1$ with the formula $\quad x_{n+1} = 3 + \dfrac{1}{x_n^2}$
to find this root correct to four decimal places.

11.4 Solution of equations where the unknown occurs in the index

Let's consider the following examples.

Example 11.8
Find x given that

$$5^x = 8.$$

We can move the unknown from the index by taking logs.

Then $\ln(5^x) = \ln 8.$

so $x \ln 5 = \ln 8.$

$\therefore \quad x = \dfrac{\ln 8}{\ln 5}.$

Rule VII Chapter 10

Leave the answer in its exact form unless an approximate value is required. Otherwise, $x = 1.2920$, correct to four decimal places, say.

Example 11.9
By first writing $y = 3^x$, solve the equation

$$3^{2x} - 3^{x+2} + 20 = 0.$$

Now if $y = 3^x$, then $3^{2x} = y^2$ and

$$3^{x+2} = 3^2.3^x = 9y.$$

Substitution into the original equation gives

$$y^2 - 9y + 20 = 0.$$

$\therefore \quad (y-5)(y-4) = 0$

so that $y = 5, 4.$

Then $3^x = 5$ or $3^x = 4$

so $x = \dfrac{\ln 5}{\ln 3}$ or $\dfrac{\ln 4}{\ln 3}.$

or use the quadratic formula

Example 11.10
Find y given that

$$7 \times 2^y = 3 \times 5^{2y}$$

Taking logs, we have

$$\ln(7 \times 2^y) = \ln(3 \times 5^{2y})$$

$\therefore \quad \ln 7 + \ln 2^y = \ln 3 + \ln 5^{2y}$

$\therefore \quad \ln 7 + y\ln 2 = \ln 3 + 2y \ln 5$

$\therefore \quad y(2\ln 5 - \ln 2) = \ln 7 - \ln 3$

Rule V, Chapter 10

Rule VII, Chapter 10

so that $y = \dfrac{\ln 7 - \ln 3}{2\ln 5 - \ln 2} \simeq 0.3355,$

correct to four decimal places.

Note that the exact value of the answer

is $\dfrac{\ln 7 - \ln 3}{2\ln 5 - \ln 2} = \dfrac{\ln\left(\frac{7}{3}\right)}{\ln\left(\frac{25}{2}\right)}.$

Exercises 11.4

1. Find x given that $3^x = 7$.

2. Given that $3^{2x+1} = 2^x$, find x.

3. Given that $3^{y+1} = 4^{y-1}$, find y.

4. By first writing $y = 2^x$, solve
$$2^{2x} - 2^{x+2} - 12 = 0.$$

5. By first writing $5^x = a$, $3^y = b$, find the values of x and y satisfying
$$3(5^x) - 3^y = 4,$$
$$5^{x+1} + 2(3^{y+1}) = 45.$$

6. Use the factor theorem to solve
$$2y^3 - 5y^2 - 9y + 18 = 0.$$
Deduce the value of x satisfying
$$2(5^{3x}) - 5^{2x+1} - 9(5^x) + 18 = 0.$$

7. Use the iterative formula
$$x_{n+1} = \ln x_n + 3$$
with starting value $x_0 = 4.5$ to find, correct to three decimal places, a value of x satisfying
$$x - \ln x - 3 = 0.$$
Deduce an approximate value of y satisfying
$$3^y = y \ln 3 + 3.$$

POST SCRIPT

Cautionary Note

The essence of the factorisation method of solving equations is to write the expression as a product of bracketed expressions.

Then ( )( )( ) = 0.

One of these factors could be equal to zero.

The roots of the equation could then be found by considering the possibility that each bracketed expression is equal to zero. Note that the presence of 0 on the right hand side is crucial.

Students should avoid the following illogical argument.

Given $2x^3 + x^2 - 15x - 18 = 0$

then $2x^3 + x^2 - 15x = 18$.

$\therefore$ $x(2x^2 + x - 15) = 18$.

Then $x = 18$

or $2x^2 + x - 15 = 18$. **X**

O.K. so far but not very useful

FAULTY LOGIC: if $ab = 18$ it does not follow that $a = 18$ or $b = 18$.

Note that
$$ab = 0 \Rightarrow a = 0 \text{ or } b = 0$$
but if $ab = c$ (where $c \neq 0$) it does **not** follow that $a = c$ or $b = c$.

Chapter 12

Some Aspects of Proof

In this chapter, we take a brief look at the use of proof in mathematics.

12.1 The need for proof

In mathematics it is tempting on the basis of checking a number of special cases to deduce that a general conjecture is true.

Example 12.1

A prime number is a number which has no factors other than itself and one. Thus 1, 2, 3, 5, 7, 11 are primes but $6 = 2 \times 3$, $9 = 3 \times 3$ are not.

Let's consider $f(n) = n^3 - 4n^2 + 7n + 1$, where n is a positive integer.

Now $f(1) = 5$, $f(2) = 7$, $f(3) = 13$, $f(4) = 29$, $f(5) = 61$, all primes.

A possible conjecture would therefore be that when n is an integer, $n^3 - 4n^2 + 7n + 1$ is a prime. Note that since the conjecture is true for n = 1, 2, 3, 4, 5 it <u>may not be true</u> for all integer values. Indeed, $f(6) = 115$ which is not a prime.

A correct proof is the only way to convince another of the truth of a conjecture. There are a number of methods of proof.

Examples of <u>direct proof</u> are the derivation of sums of arithmetic and geometric series (**P1**) and the laws of logarithms (**Chapter 10**).

Proof by <u>mathematical induction</u> occurs in the **P4** course.

In this chapter, we consider <u>proof by contradiction</u> and <u>disproof by counter-example.</u>

12.2 Proof by contradiction

Proof is concerned with the demonstration of the truth of an assertion. The essence of proof by contradiction is to assume that the assertion is false and show that the assumption leads to a contradiction. The method is illustrated by the following examples.

Example 12.2

Prove that if n^2 is even, then n is even.

Given that n^2 is even, <u>assume</u> that n is not even i.e. that n is odd.

If n is odd, $n = 2k + 1$, where k is an integer.

Then $\quad n^2 = (2k+1)^2 = 4k^2 + 4k + 1$

$\qquad = 1 + 2(2k^2 + 2k)$, which is odd.

But n^2 is even (given).

Contradiction.

The assumption is false and n is even.

Example 12.3

Show that $\sqrt[3]{2}$ is irrational, i.e. it cannot be expressed in the form $\frac{r}{s}$, where r and s are integers.

Assume that $\sqrt[3]{2} = \frac{r}{s}$, (1)

where r and s are integers having no common factors.

Any common factors can be cancelled out.

Then if $\sqrt[3]{2} = \frac{r}{s}$,

$$2 = \left(\frac{r}{s}\right)^3$$

and $r^3 = 2s^3$. (2)

Thus 2 divides r^3

or 2 divides $r \times r \times r$.

Thus, 2 divides r.

$\therefore$ $r = 2k$, (3)

where k is an integer.

Substitution from (3) into (2) gives

$$(2k)^3 = 2s^3$$

or $s^3 = 4k^3$.

Thus 2 divides s^3

In fact, 4 divides s^3

or 2 divides s × s × s.

Then 2 divides s so that

$s = 2l$, (4)

where l is an integer.

From (3) and (4), r and s have a common factor.
But r and s have no common factor (assumption).

Contradiction.

The assumption is false and $\sqrt[3]{2}$ is irrational.

Example 12.4

Given that $f(x)$ is a polynomial of degree n, show that

$$f(x) = 0$$

cannot have more than n distinct roots.

Assume that $f(x) = 0$ has more than n distinct roots. Then the equation has at least $n + 1$ roots:-

$$a_1,\ a_2, \ldots\ldots\ldots\ldots\ a_n,\ a_{n+1} \text{ (say)}.$$

Then $(x - a_1)(x - a_2) \ldots\ldots\ (x - a_n)(x - a_{n+1})$

is a factor of $f(x)$, in other words $f(x)$ has a factor of the form

$$x^{n+1} + (\ldots\ldots\ldots)x^n + \ldots\ldots,$$

a polynomial of degree $n+1$.

Thus, $f(x)$ is a polynomial of degree of at least $n+1$.

But $f(x)$ is of degree n (given).

Contradiction.

Our assumption is false and $f(x)=0$ cannot have more than n distinct roots.

Example 12.5

Use a proof by contradiction to show that if a and b are real numbers, then

$$a^2+b^2 \geq 2ab.$$

Assume that $a^2+b^2<2ab$.

Then $a^2+b^2-2ab<0$

so that $(a-b)^2<0$.

Since the square of a real number cannot be negative, $a-b$ is not a real number.

But a, b are real numbers (given) so that $a-b$ is a real number.

Contradiction.

Our assumption is false and $a^2+b^2 \geq 2ab$.

Example 12.6

Use a proof by contradiction to show that

if $(x+1)(x-3)<0$ then $-1<x<3$.

Assume that if $(x+1)(x-3)<0$

then $x \leq -1$ or $x \geq 3$.

There are two cases to be considered.

$x \leq -1$

Let $x=-1-a$ where $a \geq 0$.

Then $(x+1)(x-3)=(-1-a+1)(-1-a-3)$

$=(-a)(-1-a-3)$

$=(-a)(-a-4)$

$=a(a+4) \geq 0$.

$a \geq 0$

But $(x+1)(x-3)<0$ (given).

Contradiction.

The assumption is false and $x>-1$ (i)

$x \geq 3$

Let $x=3+b$ where $b \geq 0$.

Then $(x+1)(x-3)=(1+3+b)(3+b-3)$

$=(4+b)b \geq 0$.

$b \geq 0$

But $(x+1)(x-3)<0$.

Contradiction.

The assumption is false and $x<3$. (ii)

Combining statements (i) and (ii), we conclude that

if $(x+1)(x-3)<0$ then $-1<x<3$.

Example 12.7

Prove by contradiction that if x is real and $x > 0$ then

$$x + \frac{4}{x} \geq 4.$$

Assume that $$x + \frac{4}{x} < 4.$$

Then multiplying by x and noting that $x > 0$, we obtain

$$x^2 + 4 < 4x$$

or $$x^2 - 4x + 4 < 0.$$

$\therefore$ $$(x-2)^2 < 0.$$

Then $x - 2$ is not real since the square of a real number is never negative.
But x is real (given) and 2 is real and therefore $x - 2$ is real.

Contradiction.

Our assumption is false and

$$x + \frac{4}{x} \geq 4.$$

Proof by contradiction may sometimes be used to prove that a function is one-one.

Example 12.8

Note that a function f is one-one if given $f(a) = f(b)$, then $a = b$.
Show that the function f defined by

$$f(x) = x^2 - 2x + 3 \text{ for } x > 1$$

is one-one.

Assume that f is not a one-one function; in other words there exist a and b

such that $f(a) = f(b)$

and $a \neq b$.

Under the assumption,

$$a^2 - 2a + 3 = b^2 - 2b + 3$$

or $$a^2 - b^2 - 2a + 2b = 0$$

$\therefore$ $$(a-b)(a+b) - 2(a-b) = 0.$$

$\therefore$ $$(a-b)(a+b-2) = 0. \qquad (1)$$

Since $a \neq b$, (assumption)

$$a - b \neq 0$$

and the non zero factor $a - b$ may be cancelled from (1).

Then $$a + b - 2 = 0. \qquad (2)$$

But from the definition of the domain of the function,

$$a > 1,$$
$$b > 1$$

and so $$a + b > 2$$

or $$a + b - 2 > 0. \qquad (3)$$

Now (3) contradicts (2).

Our assumption is false and f is a one-one function.

Exercises 12.1
Use proof by contradiction to prove the following.

1. Show that if n^3 is odd, then n is odd.

2. Show that if $2n^2 + n$ is even, then n is even.

3. Show that $\sqrt{3}$ is irrational.

4. Show that if x is real and $x > 0$, then

$$x + \frac{1}{x} \geq 2 .$$

5. Show that if $(x-2)(x-5) \geq 0$, then

$$x \leq 2 \quad \text{or} \quad x \geq 5 .$$

6. Show that if x and y are real, then

$$x^2 + 4y^2 \geq 4xy .$$

7. Show that if $f(x)$ is a cubic, then

$$f(x) = 0$$

cannot have more than 3 distinct roots.

8. Show that the function defined by

$$f(x) = x^2 + 4x + 5 \quad \text{for } x \leq -2 \text{ is one-one.}$$

9. Show that the function defined by

$$f(x) = x^2 + 6x + 9 \quad \text{for } x \geq 0 \text{ is one-one.}$$

12.3 Disproof by counter-example

Many conjectures in Mathematics involve the word 'all'. To show that a conjecture involving 'all' is false, it is sufficient to show that the conjecture is false in just <u>one</u> case. This approach is known as disproof by counter-example.

Example 12.9
A student asserts that

$$\sin(\theta_1 + \theta_2) = \sin\theta_1 + \sin\theta_2$$

for all angles θ_1, θ_2.
Use a counter-example to show that this assertion is false.

Now if $\theta_1 = 30°,\ \theta_2 = 60°$ (for example)

$$\sin(\theta_1 + \theta_2) = \sin(30° + 60°) = \sin 90° = 1$$

and $\sin\theta_1 + \sin\theta_2 = \sin 30° + \sin 60° = \frac{1}{2} + \frac{\sqrt{3}}{2} \neq 1$.

Thus it is not true that

$$\sin(\theta_1 + \theta_2) = \sin\theta_1 + \sin\theta_2$$

for all angles θ_1, θ_2.

Example 12.10
Give a counter-example to disprove the following:-

if $f'(0) = 0$ then $f(x)$ has a maximum or minimum at $x = 0$.

A suitable counter-example is

$$f(x) = x^3.$$

Then $f'(0) = 0$ but there is not a maximum or minimum at $x = 0$ but a stationary point of inflexion.

Example 12.11
Use a counter-example to show that the function f defined by $f(x) = 3 + 4x - x^2$ for all x is not one-one.

To show f is not one-one we require values a and b such that

$$f(a) = f(b)$$

with $a \neq b$.
Now $f(x) = 3 + 4x - x^2 = 7 - (x-2)^2$.
Then $a = 0, b = 4$ will do the trick,
since $f(0) = 7 - (-2)^2 = 3$
and $f(4) = 7 - (4-2)^2 = 3$.
Thus f is not one-one.

Exercises 12.2
Use counter-examples to show that the following statements are false.

1. $\cos 2\theta = 2\cos\theta$ for all values of θ.

2. $\ln(x + y) = \ln x + \ln y$ for all $x, y > 0$.

3. For all real values of x and y,
   if $x > y$ then $x^2 > y^2$.

4. If, in the quadratic equation
   $ax^2 + bx + c = 0$,
   a, b, c are real and b is negative, then the roots are negative.

5. If $f'(0) = 0$ and $f''(0) = 0$, then $f(x)$ has a stationary point of inflexion at $x = 0$.

6. For any real numbers n, p with $p > 0$,
   $\ln(p^n) = (\ln p)^n$.

Revision Paper 1

1. Solve the inequalities

(a) $| 5x - 3 | < 7$,

(b) $(x - 3)(x - 2) > 12$.

2. Write down the binomial expansion for $(1 + x)^{10}$ in ascending powers of x as far as the term in x^3.

Given that

$$(1 + ax + bx^2)(1 - 2x)^{10} \equiv 1 - 23x + 242x^2 + cx^3 + \ldots,$$

(a) find a and b,

(b) find c.

3. Given a function f defined by

$$f(x) = 1 - \frac{1}{2x^2 + 3}, \qquad x > 0$$

(a) find $f'(x)$ and deduce that f is an increasing function,

(b) give the range of f,

(c) derive an expression for $f^{-1}(x)$.

4. If $\ln x = 1.3614$, $\ln y = 2.1469$, $\ln z = 0.6158$ evaluate the following, using your calculator for +, ×, ÷ only and showing all your working.

(a) $\ln(xy)$ (b) $\ln\left(\dfrac{x^2\sqrt{y}}{z^{\frac{3}{2}}}\right)$.

5. (a) Integrate the following with respect to x.

(a) e^{3-2x} (b) $\dfrac{1}{(7x-9)}$ (c) $\dfrac{1}{(2-5x)^2}$

6. (a) Differentiate $(x^2 + 1)\ln(x^2 + 1)$ with respect to x.

(b) Show that $\dfrac{d}{dx}\left(\dfrac{e^x + x}{x+1}\right) = \dfrac{xe^x + 1}{(x+1)^2}$. Hence evaluate $\displaystyle\int_0^1 \frac{xe^x}{(x+1)^2}\,dx$.

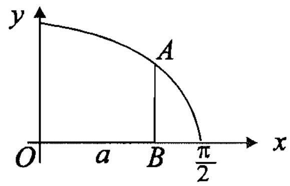
7. Show that the circle which has the line joining two points (1, 1) and (3, 5) as a diameter is given by

$$x^2 + y^2 - 4x - 6y + 8 = 0.$$

Show that the equation of the tangent to the circle at (0, 2) is

$$y + 2x = 2.$$

8. A is the point $(a, \cos a)$ on the curve $y = \cos x$ and AB is perpendicular to Ox as shown.



The area of triangle AOB is one quarter of the area below the curve between the lines $x = 0$, $x = a$ and the x-axis.

(a) Show that $a = \dfrac{1}{2}\tan a$.

(b) Show that there is a root of the equation between $\dfrac{\pi}{3}$ and $\dfrac{5\pi}{12}$.

(c) Use the iterative process

$$a_{n+1} = \tan^{-1}(2a_n)$$

with $a_0 = 1.16$ to find the value of a correct to three decimal places.

9. Prove by contradiction that the equation

$$x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0,$$

where a_1, a_2, a_3, a_4 are constants, cannot have more than four distinct roots.

Revision Paper 2

1. Given that $x + 1$ and $x - 2$ are factors of

$$x^4 + ax^3 - 7x^2 + bx + 6,$$

find all the factors of the expression.

2. Write down the binomial expansion of $(1 + 2x)^4$.
Solve the equation

$$(1 + 2x)^4 + (1 - 2x)^4 - 34 = 0.$$

3. (a) Use a counter-example to show that the function f defined by

$$f(x) = x^2 + 6x + 11 \quad \text{for all x}$$

is not a one-one function.

(b) Use proof contradiction to show that the function g defined by

$$g(x) = x^2 + 6x + 11 \text{ for } x > -3$$

is a one-one function.

4. Find x in the following cases.

(a) $5^{2x} = 3(2^x)$, correct to four decimal places. (b) $4^{2x+1} + 16 = 65 \times 4^x$.

5. (a) Given $f(x) = \dfrac{x^2 - a}{x^2 + a}$, find $f'(x)$.

Given $f'(1) = 1$, find the value of a.

(b) Differentiate the following with respect to x.

(i) $\ln(e^x + x)$ (ii) $x^2(1 + x)^{12}$ (iii) $\sin 4x$

6. Integrate the following with respect to x.

(a) $\displaystyle\int \frac{3}{4x+5}\,dx$ (b) $\displaystyle\int \frac{2}{(2x+1)^2}\,dx$ (c) $\displaystyle\int \sin(3x+5)\,dx$

7. A circle has equation

$$x^2 + y^2 - 2x - 4y - 5 = 0.$$

(a) Find the coordinates of the centre of the circle.

(b) Find the radius of the circle.

(c) The line $y = 2x + 5$ intersects the circle at points A and B. O is the centre of the circle. Show that BO is perpendicular to AO and find the area of triangle ABO.

8. Use the trapezium rule with five ordinates to find an approximate value for

$$\int_0^1 \frac{dx}{\sqrt{1+x^2}},$$

giving your answer correct to three decimal places.

Revision Paper 3

1. (a) Factorise $5x^3 - 4x^2 - 11x - 2$.

(b) Find y if $5^{3y+1} - 4 \times 5^{2y} - 11 \times 5^y - 2 = 0$.

2. The functions f and g are defined by

$$f(x) = \frac{1}{\sqrt{x-3}} \qquad (x > 3)$$

and $$g(x) = 3x^2 - 3 \qquad (x > 0).$$

(a) Derive an expression for $gf(x)$.

(b) Derive an expression for $g^{-1}(x)$ and sketch the graphs of $g(x)$ and $g^{-1}(x)$ on the same diagram.

3. (a) Given that $p = e^{\ln p}$, where p is real and positive,

show that $\ln(p^n) = n \ln p$.

(b) A student claims that if x, y are both real and positive, then

$$\ln(xy) = (\ln x)(\ln y).$$

Use a counter-example to show that this assertion is false.

(c) Given that

$$5 \times 2^{x+1} = 3 \times 5^{x+2},$$

show that

$$x = \frac{\ln 2 - \ln 15}{\ln 5 - \ln 2}.$$

4. The circle C is given by the equation

$$x^2 + y^2 - 6x - 4y + 4 = 0.$$

(a) Write down the coordinates of the centre of the circle.

(b) Show that the distance of the centre of the circle from the y-axis is equal to the radius. What does this result indicate concerning the y-axis and the circle?

5. (a) Differentiate $\dfrac{e^{2x}}{3x-5}$ with respect to x.

(b) Differentiate $x^2 \ln x$ with respect to x. Hence show that

$$\int_1^2 x \ln x \, dx = 2 \ln 2 - \frac{3}{4}.$$

6. (a) A chord is joining the points at which $x = 0$ and $x = \frac{\pi}{6}$ on the curve $y = \cos 2x$.

Find the values of x in the range 0 to $\frac{\pi}{2}$ at the points where the tangents are parallel to the chord.

(b) Show that $\displaystyle\int_0^{\frac{\pi}{4}} (\sin 3x - \sin 5x) dx = \frac{\sqrt{2}}{15} + \frac{2}{15}$, given that $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$.

7. Show that there is a root of the equation

$$x = 2\sin x$$

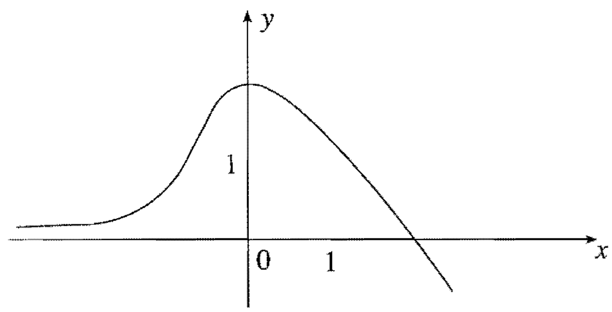
between $\frac{\pi}{2}$ and π.

Use the iterative formula

$$x_{n+1} = \frac{2(-x_n \cos x_n + \sin x_n)}{1 - 2\cos x_n}$$

with starting value $x_0 = 2$ to find the root correct to two decimal places.

8.



The sketch shows the graph of $y = f(x)$. The curve passes through the point (1,0) and has a maximum point at (0, 1).

Sketch on separate diagrams graphs of

(a) $y = f(x) + 1$

(b) $y = f(x+1)$

(c) $y = f\left(\frac{x}{2}\right)$.

Revision Paper 4

1. Write down the binomial expansion for $(a + b)^4$.

Find the term independent of x in the binomial expansion of $\left(3x - \frac{2}{x}\right)^4$.

2. Prove by contradiction that if

$$x^2 - 6x + 8 < 0$$

then $2 < x < 4$.

3. If

$$f(x) = \frac{\sqrt{x^2 - 1}}{x}$$

with domain $x \geq 1$,

(a) find an expression for $f^{-1}x$,

(b) state the domain and range of f^{-1},

(c) explain why the function of ff cannot be formed.

4. Solve the following equations

(a) $3^x = e^{2x+1}$

(b) $3\ln 2x = 1 + \ln x$.

5. A circle C has equation

$$x^2 + y^2 + 4x - 4y - 8 = 0.$$

The straight line with equation $x + y = 4$ cuts C at two points A and B.

(a) Find the coordinates of A and B.

(b) If O is the centre of C, find the area of triangle AOB.

6. Differentiate the following with respect to x.

(a) $\dfrac{e^x + 1}{e^x + 2}$ (b) $\ln(\sin x)$ (c) $x^2(1 + x^3)^{20}$

7\. (a) Integrate the following with respect to x.

(i) $\dfrac{(e^x + e^{-x})^2}{e^x}$ (ii) $\dfrac{1}{3x + 2} + \dfrac{2}{(4x + 7)^2}$

(b) Find the area between the curve $y = \sin 2x$, the x-axis and the lines $x = 0$, $x = \dfrac{\pi}{2}$.

8. Use the trapezium rule with five ordinates to find an approximate value for

$$\int_2^3 \sqrt{x - \frac{1}{x}}\,dx,$$

giving your answer correct to two decimal places.

Revision Paper 5

1. (a) Express the polynomial

$$2x^4 - 3x^3 - 2x^2 + x - 2$$

as a product of two linear factors and a quadratic factor.

(b) Use the result of (a) to show that there is only one value of θ in the range 0° to 360° satisfying $2\sin^4\theta - 3\sin^3\theta - 2\sin^2\theta + \sin\theta - 2 = 0$.

2. Sketch the graphs of (a) $y = \ln x$, (b) $y = \ln(2x)$ (c) $y = 2\ln x + 3$,
showing the points (i) where $y = 0$,
(ii) where $x = e$.

3. The functions f and g are defined by

$$f(x) = \frac{2}{x-1}, \quad x \neq 1$$

$$g(x) = x^2 + 2 \quad \text{for all } x.$$

(a) Find the values of x for which
$f(x) = x$.

(b) State the range of g.

(c) Find $fg(x)$ and state the range of fg.

(d) State whether the inverse of g exists, giving a reason for your answer.

4. (a) Given that x and y are real and positive, show that

$$\ln(xy) = \ln x + \ln y.$$

(b) Solve the equation

$$\ln(x^2 - 10) = \ln x + 2\ln 3.$$

5. Differentiate the following with respect to x, simplifying your answers as far as possible.

(a) $\dfrac{2e^x}{e^{2x}+1}$ (b) $x^3\cos 3x$ (c) $\dfrac{1}{\sqrt{4x^2+5}}$

6. (a) Find $\displaystyle\int\left(\frac{1}{4x+5} + \frac{3}{(3x+2)^3}\right)dx.$

(b) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}}(\cos 2x + \sin x)\,dx.$

(c) Find $\displaystyle\int\frac{e^{3x}+1}{e^{2x}}dx.$

7. Sketch the graphs of $y = \cos 2\theta + 1$ and $y = 4\theta$ for $0 \leq \theta \leq \pi$ and hence show there is a root of

$$\cos 2\theta + 1 = 4\theta$$

between 0 and $\frac{\pi}{4}$. Show further that the root lies between 0.4 and 0.5.

Use the iterative formula

$$\theta_{n+1} = \frac{1}{4}(1 + \cos 2\theta_n)$$

with starting value $\theta_0 = 0.4$ to find the root correct to three decimal places.

8. Use proof by contradiction to show that $\sqrt{5}$ is irrational.

Revision Paper 6

1. (a) Write down the binomial expansion of $(a + bx)^8$ as far as the term in x^3.

   (b) In the binomial expansion for $(a + bx)^8$,

   (i) the term in x has coefficient 64,

   (ii) the coefficient of x^2 is equal to the coefficient of x^3.

   Show that $a = \pm\sqrt{2}$.

2. Solve the inequalities

   (a) $|2x + 3| > 5$,

   (b) $3x^2 - 4x + 3 \le 2x^2 - 3x + 5$.

3. Find the values of x and y satisfying the simultaneous equations

   $3 \times 5^x + 2 \times 7^y = 13$,

   $7 \times 5^x + 3 \times 7^y = 20$.

4. The quadratic expression

   $f(x) = ax^2 + bx + c$

   is such that

   (a) when it is divided by $x - 1$, the remainder is 3,

   (b) when it is divided by $x + 1$, the remainder is 7,

   (c) $f(0) = 1$.

   Find $f(x)$.

5. The function f is defined by

   $$f(x) = x + \frac{1}{x} \text{ for } x > 1$$

   (a) State the range of f.

   (b) Show that f is an increasing function.

   (c) Find an expression for $f^{-1}(x)$ and state the domain and range of f^{-1}.

   (d) Sketch the graphs of f and f^{-1} on the same axes.

6. (a) Differentiate $\sin x \cos x$ with respect to x.

   (b) Differentiate $xe^{2x}dx$. Hence evaluate

   $$\int_0^1 xe^{2x}dx.$$

7. Integrate the following with respect to x.

   (a) e^{4x-9} (b) $\cos(5x + 7)$ (c) $\dfrac{1}{(3 + 2x)}$

8. (a) Use a counter-example to show that the following assertion is false:-

   $\tan(\theta_1 - \theta_2) = \tan\theta_1 - \tan\theta_2$ for all values of θ_1, θ_2.

   (b) Use proof by contradiction to show that if $4n^2 + n + 1$ is even, then n is odd.

ANSWERS

Chapter 1
Exercises 1.1

(a) $12 > 9$ (b) $4 < 7$ (c) $x \geq y$ (d) $m > 0$ (e) $p \geq 0$

Exercises 1.2

1. (i) $x > 4$ (ii) $x > -4$ (iii) $x > \frac{4}{7}$ (iv) $x < \frac{3}{5}$

(v) $x > -12$ (vi) $x \leq \frac{5}{3}$ (vii) $x > -21$ (viii) $x \leq 14$

2. (i) $x < -5$ or $x > -1$ (ii) $4 < x < 5$ (iii) $x \leq -7$ or $x \geq 1$

(iv) $-12 \leq x \leq -6$ (v) $x < \frac{5-\sqrt{17}}{4}$ or $x > \frac{5+\sqrt{17}}{4}$

(vi) $\frac{8}{5} < x < 2$ (vii) $x < -\frac{8}{5}$ or $x > 2$ (viii) $\frac{5-\sqrt{33}}{4} \leq x \leq \frac{5+\sqrt{33}}{4}$

(ix) $x \leq \frac{7-\sqrt{13}}{6}$ or $x \geq \frac{7+\sqrt{13}}{6}$ (x) $\frac{-5-\sqrt{41}}{8} \leq x \leq \frac{-5+\sqrt{41}}{8}$

(xi) $-1 < x < 2$

3. (i) $k \geq -\frac{4}{3}$ (ii) $-\sqrt{48} < k < \sqrt{48}$ (iii) $k \leq 0$ or $k \geq 4$

(iv) $k < -\frac{17}{8}$ (v) $k \leq -\frac{11}{12}$

Exercises 1.3

1. (i) 2 (ii) 1 (iii) 0 (iv) 24

Exercises 1.4

1. (i) $-16 < x < 2$ (ii) $x < -\frac{3}{2}$ or $x > \frac{9}{2}$ (iii) $-\frac{1}{4} < x < \frac{11}{4}$

(iv) $x \leq \frac{1}{2}$ or $x \geq \frac{5}{2}$

2. (i) $x = -3, 7$ (ii) $x = -2, 5$ (iii) $x = -4, 9$

(iv) $x = 6 \pm \sqrt{34},\ 6 \pm 2\sqrt{7}$ (v) $y = \frac{-2 \pm \sqrt{10}}{2}, \frac{-2 \pm \sqrt{2}}{2}$

Exercises 1.5

1. (a) $(-3, \infty)$ (b) $(-\infty, 6]$ (c) $[9, \infty)$ (d) $(-\infty, -4)$ (e) $[-3, 21)$
(f) $[9, 12)$ (g) $(-5, 20]$ (h) $(-\infty, -30] \cup [-20, \infty)$

2. (i) $(-\infty, -5) \cup (-1, \infty)$ (ii) $(4, 5)$ (iii) $(-\infty, -7] \cup [1, \infty)$ (iv) $[-12, -6]$

Chapter 2

Exercises 2.1

1. (i) $(x+3)(x-6)+16$ (ii) $(x-5)(x^2+2x+14)+65$
(iii) $(2x+1)(x^2-4x+5)-8$ (iv) $(6x+5)(2x^3-3x^2+6x-5)+26$
(v) $(4x^2-3x+2)(3x^2+2x+3)+5x-13$

2. (i) x^2+4x+1 (ii) x^2-1 (iii) $x^2-2x+1=(x-1)^2$

3. $(x-5)(x-3)(x+4)$

4. $(x-2)(x-3)(x+3)(x-4)$

Exercises 2.2

1. (i) -4 (ii) 7 (iii) 0 (iv) -3

2. -7 **3.** $2, -1$

4. (i) $(x+1)(x-2)^2$ (ii) $(x-1)(x^2-x-1)$
(iii) $(x-1)(x+2)(x^2-x+2)$ (iv) $(x-2)(x^2-x-3)$

5. -14 **6.** $-15, 26$

7. $0, -2$; $(x+1)^2(x-2)$ **9.** 3

Chapter 3

Exercises 3.1

1. (i) $a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$
(ii) $a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$
(iii) $a^8+8a^7b+28a^6b^2+56a^5b^3+70a^4b^4+56a^3b^5+28a^2b^6+8ab^7+b^8$

2. $1+6y+12y^2+8y^3$ **3.** $x^4+12x^3y+54x^2y^2+108xy^3+81y^4$

4. $1-9y+27y^2-27y^3$ **5.** (i) $17+12\sqrt{2}$ (ii) $9\sqrt{3}+11\sqrt{2}$ (iii) 32

Exercises 3.2

1. (i) 5040 (ii) 2520 **2.** 60 **3.** 720 **4.** 120

5. 120 ; 20 **6.** 722 **7.** 70 **8.** 35 **9.** $\dfrac{n!}{3!(n-3)!}$

Exercises 3.3

1. (i) 362880 (ii) 165765600 (iii) 53130 (iv) 15504

2. 60480 **3.** 360 ; 60 **4.** 5 **5.** 4845 ; 969 **6.** 1906884

Exercises 3.4

1. (i) $1+10z+40z^2+80z^3+80z^4+32z^5$

(ii) $x^4-8x^3y+24x^2y^2-32xy^3+16y^4$ (iii) $x^3+3x+\dfrac{3}{x}+\dfrac{1}{x^3}$

(iv) $8y^3-12y^2z+6yz^2-z^3$

2. (i) $1+12x+66x^2$ (ii) $1-28y+364y^2$ (iii) $p^{16}+16p^{15}q+120p^{14}q^2$
(iv) $1+5x+\dfrac{45x^2}{4}$ (v) $256-3072x+16128x^2$ (vi) $x^{22}+11x^{18}+55x^{14}$

3. $-\binom{20}{17}2^{17}x^{17}y^3$ or $-149422080x^{17}y^3$ **4.** $-960x^3$ **5.** 0.8508

6. 1.083 **9.** ± 2

10. $\frac{5}{48}$ **11.** 8 **12.** $\pm\frac{1}{2}$

Chapter 4
Exercises 4.1

1. $2\ ;\ \frac{5}{2}\ ;\ -2\ ;\ a+\frac{1}{a}$ **2.** No **4.** $-1, -2$

Exercises 4.2

1. (a) $(-\infty, 2]$ (b) $(-\infty, -3)\cup(-3, \infty)$ (c) $(0, \infty)$
(d) $(-2, 1)\cup(1, \infty)$ (e) $(-\infty, -1)\cup(1, \infty)$ (f) $(-\infty, 1)\cup(1, 3)\cup(3, \infty)$
(g) $(-\infty, -3)\cup\left(\frac{1}{2}, \infty\right)$ (h) $[-2, 2]$ (i) $(-\infty, 9]\cup(16, \infty)$

Exercises 4.3

1. (a) $[-1, 4]$ (b) $(4, 8)$ (c) $(1, 3]$ (d) $(0, \infty)$ (e) $[0, 25]$
2. (a) No (b) No (c) Yes (d) No (e) No (f) Yes (g) Yes

Exercises 4.4

1. (a) One-one (b) Not one-one (c) One-one (d) Not one-one
(e) Not one-one (f) Not one-one (g) One-one (h) Not one-one

2. (a) $f^{-1}(x) = \frac{x-3}{4}\ ;\ [-1, 23]\ ;\ [-1, 5]$
(c) $f^{-1}(x) = -3+\sqrt{9+x}\ ;\ [-9, 27]\ ;\ [-3, 3]$
(g) $f^{-1}(x) = \frac{1}{x}\ ;\ (0, \infty)\ ;\ (0, \infty)$

3. (a) $f^{-1}(x) = \frac{1}{x-1}\ ;\ (1, \infty)\ ;\ (0, \infty)$
(b) $f^{-1}(x) = -2+\sqrt{x-3}\ ;\ (7, \infty)\ ;\ (0, \infty)$
(c) $f^{-1}(x) = -3+\sqrt{4+x}\ ;\ (-4, \infty)\ ;\ (-3, \infty)$
(d) $f^{-1}(x) = x^2-2\ ;\ [0, \infty)\ ;\ [-2, \infty)$
(e) $f^{-1}(x) = \frac{1-2x^2}{x^2}\ ;\ (0, \infty)\ ;\ (-2, \infty)$
(f) $f^{-1}(x) = -1+\sqrt{4+x^2}\ ;\ (0, \infty)\ ;\ [1, \infty)$

4. $(-\infty, -1)\cup(-1, \infty)\ ;\ f^{-1}(x) = \frac{1-x}{x}\ ;\ (-\infty, 0)\cup(0, \infty)\ ;\ (-\infty, -1)\cup(-1, \infty)$

5. $f^{-1}(x) = -\sqrt{\frac{1-x}{x}}\ ;\ (0, 1)\ ;\ (-\infty, 0)$

6. $(-\infty, -3)\cup(-3, \infty)\ ;\ (-\infty, \infty)\ ;\ f^{-1}(x) = \frac{1+3x}{2-x}$

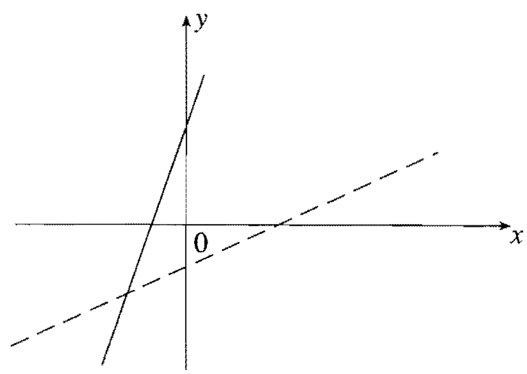
7. $[0, 2]\ ;\ [0, 2]\ ;\ f^{-1}(x) = \sqrt{4-x^2}$

Exercises 4.5

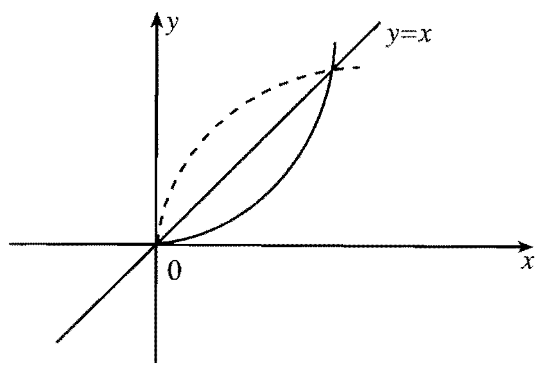
In the following, the broken line graphs relate to the inverse functions.

1. $f^{-1}(x) = \dfrac{x-2}{3}$ for all x.

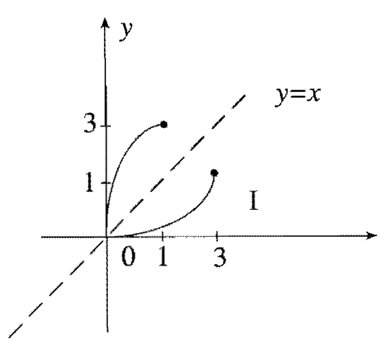
In questions 1 and 2, the broken lines indicate the inverse functions.



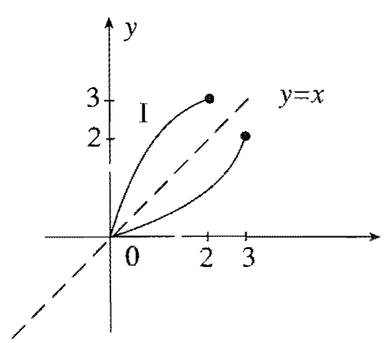
2. $f^{-1}(x) = \sqrt{x}$ for $x \geq 0$.



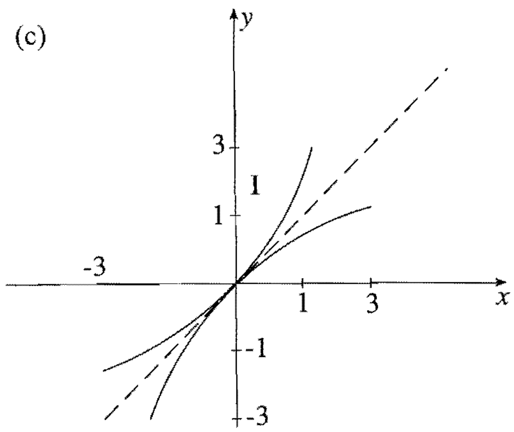
3. (a)



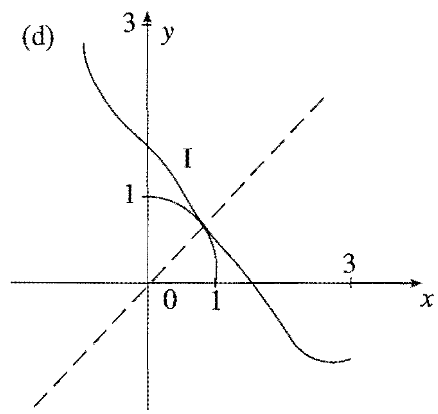
(b)



(c)



(d)



I denotes the inverse in the above cases.

Exercises 4.6

1. fg(x) = $9x^2 - 12x + 6$; gf(x) = $3x^2 + 4$
Domains and ranges are $(-\infty, \infty)$, $(-\infty, \infty)$ in both cases.

2. fg(x) = $6x - 2$; gf(x) = $6x - 11$
Domains and ranges are $(-\infty, \infty)$, $(-\infty, \infty)$ in both cases.

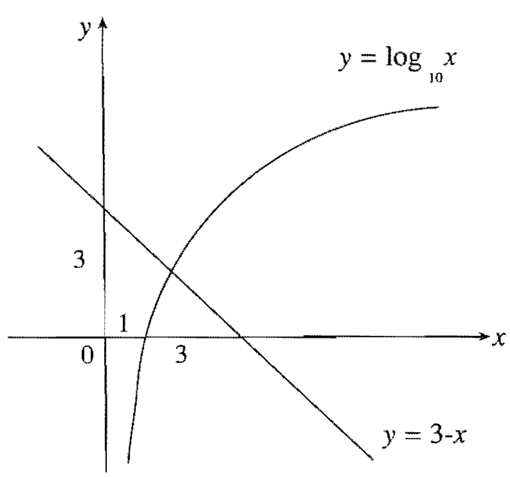
3. fg doesn't exist ; gf(x) = $2x$ with domain $(0, \infty)$ and range $(0, \infty)$.

4. (a) gf doesn't exist because the range of f ([0, 19]) is not contained in the domain of g ((4, 20)).
(b) fg exists because the range of g ((0, 4]) is contained in the domain of f ([0, 4]).
(c) fh exists because the range of h $\left(\left(\frac{2}{225}, 2\right)\right)$ is contained in the domain of f ([0, 4]).
(d) hf doesn't exist because the range of f ([3, 19]) is not contained in the domain of h ((1, 15]). Similarly, (e), (f), gh and hg do not exist.

Chapter 5
Exercises 5.1

1. (i, C) (ii, D) (iii, I) (iv, F) (v, H) (vi, G) (vii, B) (viii, A)
(ix, E) (x, M) (xi, K) (xii, L) (xiii, J)

2.



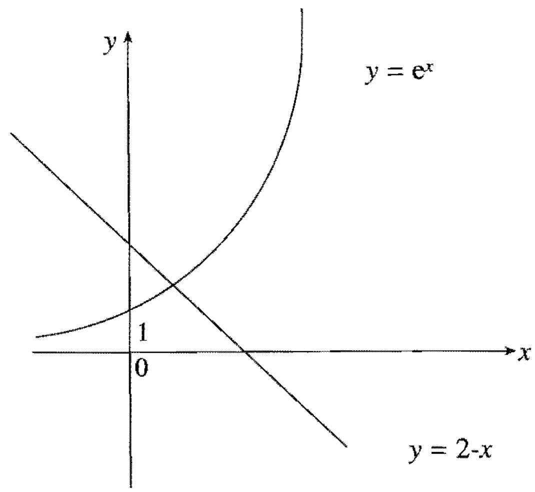
Where graphs intersect

$$3 - x = \log_{10} x$$

or $$x + \log_{10} x - 3 = 0.$$

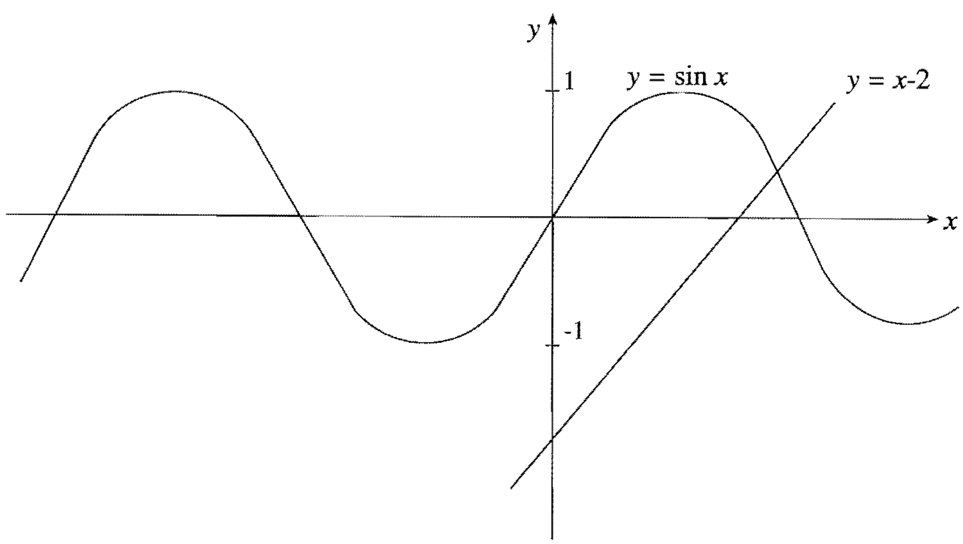
Graphs intersect at one point. Therefore, the equation has only one root.

3.



The graphs intersect at only one value of x, which is positive. Thus the equation has only one root which is positive.

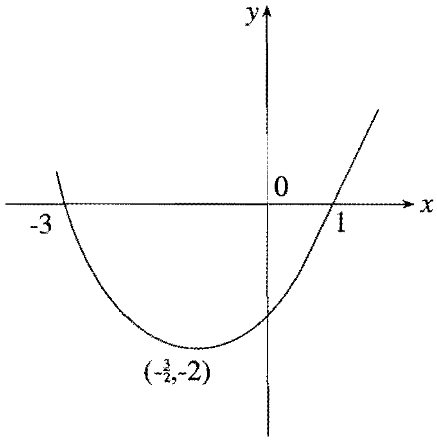
4.



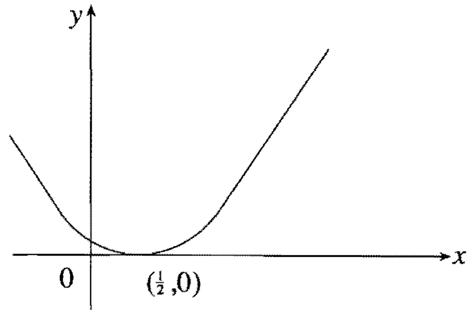
Graphs intersect at one point and, therefore, the equation has only one root.

Exercises 5.2 (drawings are not given to scale.)

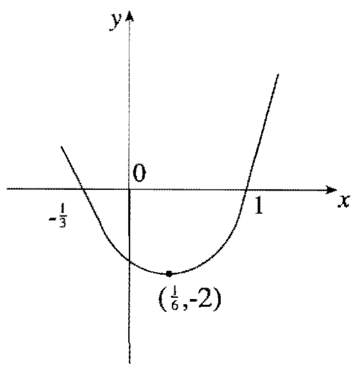
1 (a)



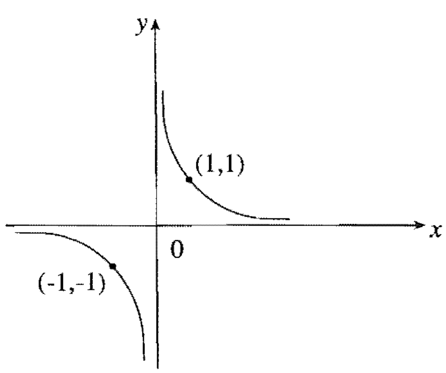
(b)



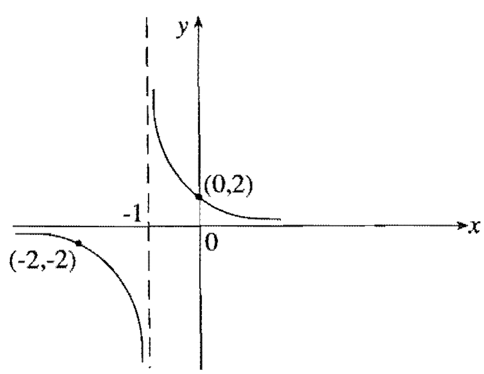
(c)



2.

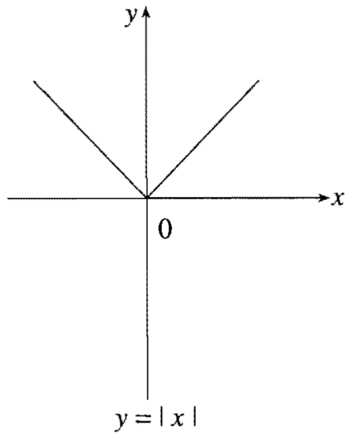


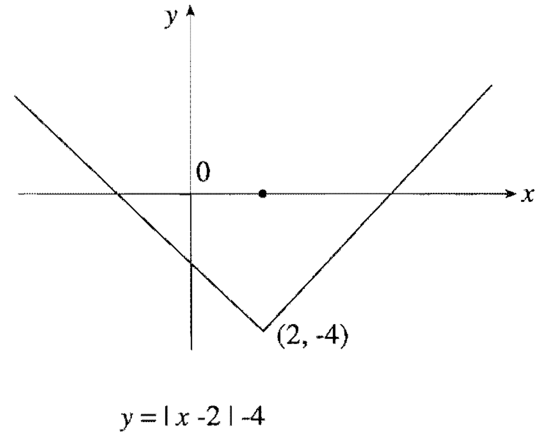
Equation $y = \dfrac{1}{x}$



Equation $y = \dfrac{2}{x+1}$

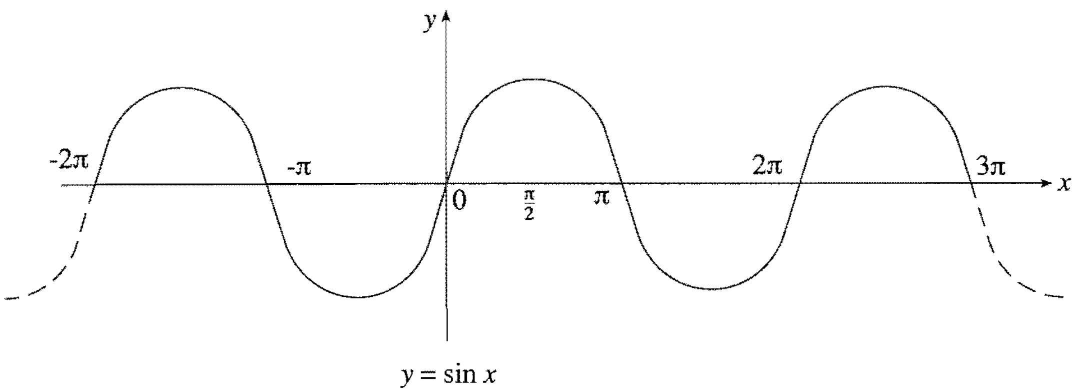
3.

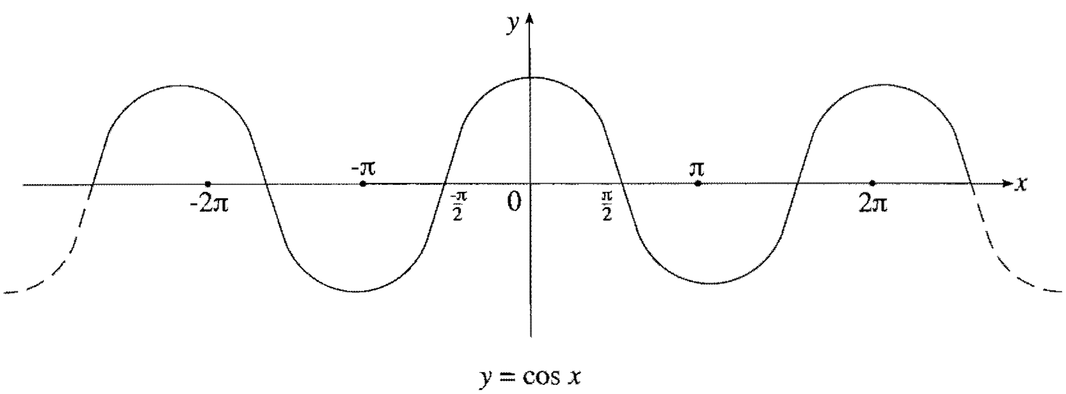




Transformations $(x, y) \longrightarrow (x+2, y)$, $(x, y) \longrightarrow (x, y-4)$.

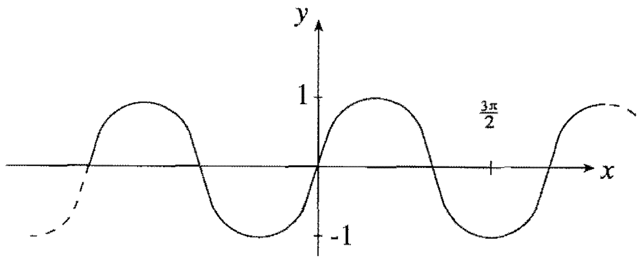
4.



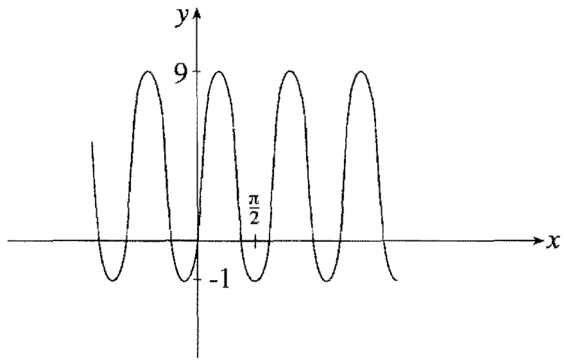


The graph $y = \sin\left(x + \frac{\pi}{2}\right)$ is obtained from $y = \sin x$ by an x translation of $\frac{\pi}{2}$ to the left, this resulting in the graph $y = \cos x$.

5.

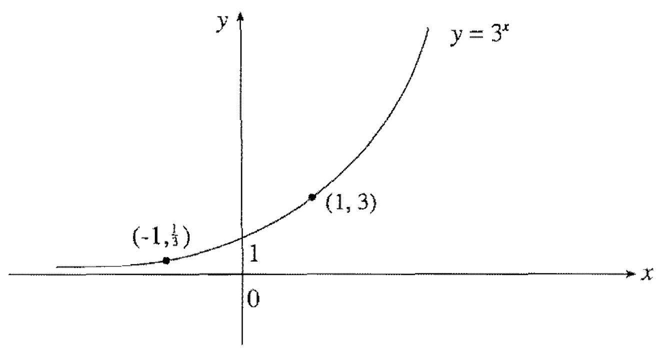


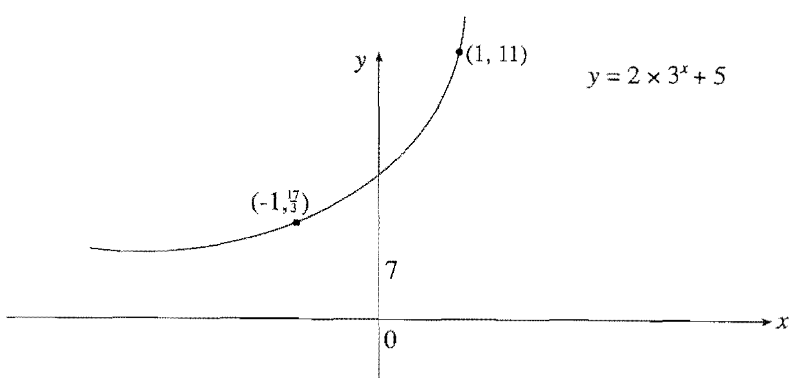
$y = \sin x$



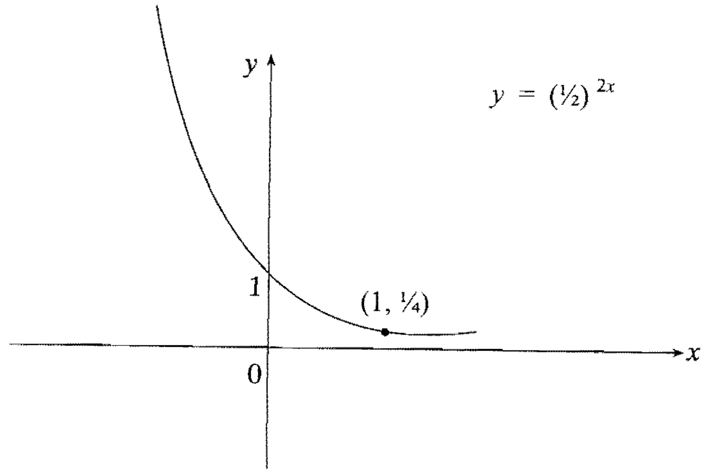
In $y = \sin x$, the peaks and troughs occur at $x = (2k+1)\frac{\pi}{2}$, (k any integer), with values at ± 1. For $y = 5\sin 3x + 4$, the peaks and troughs occur more frequently at $x = (2k+1)\frac{\pi}{6}$, with values 9 and -1.

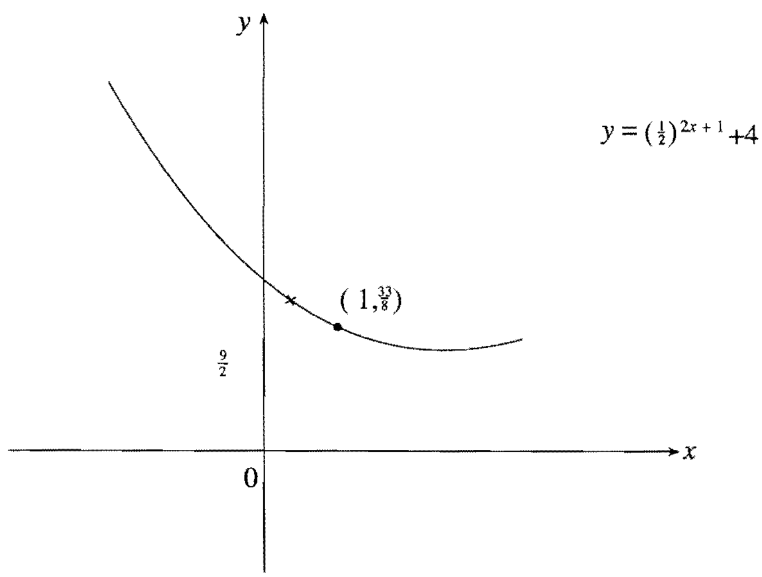
6.



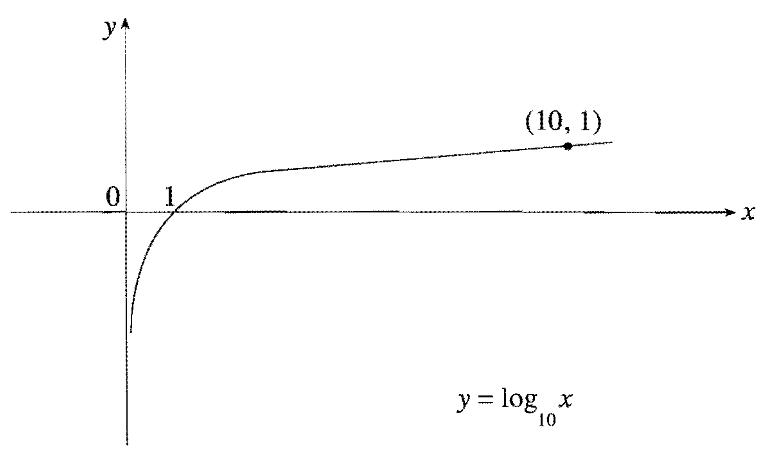


7.

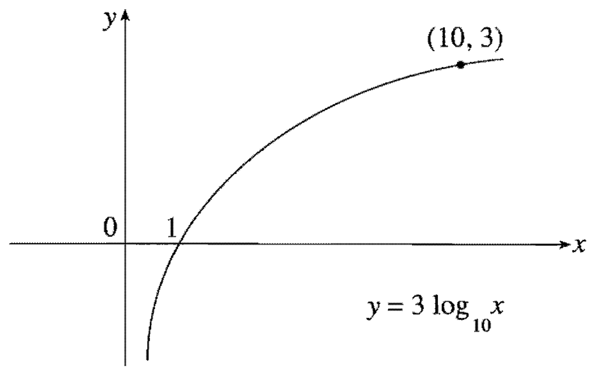




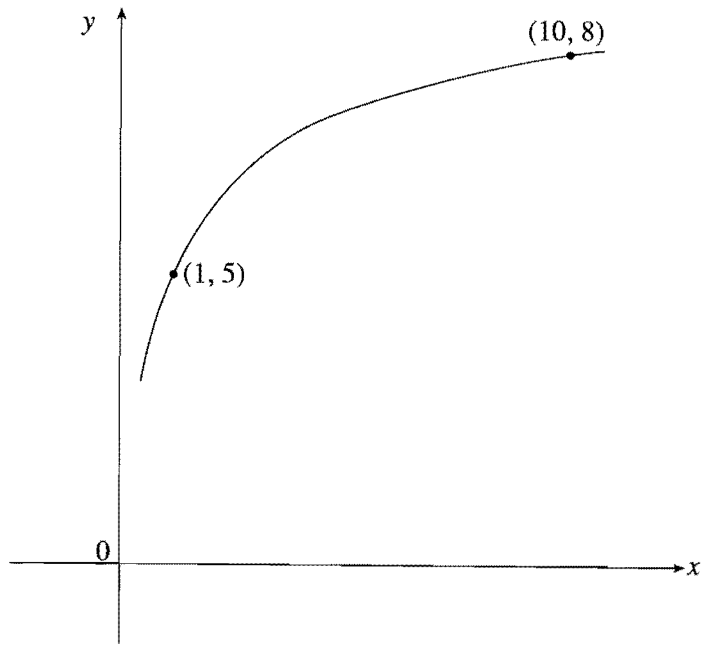
8. (i)



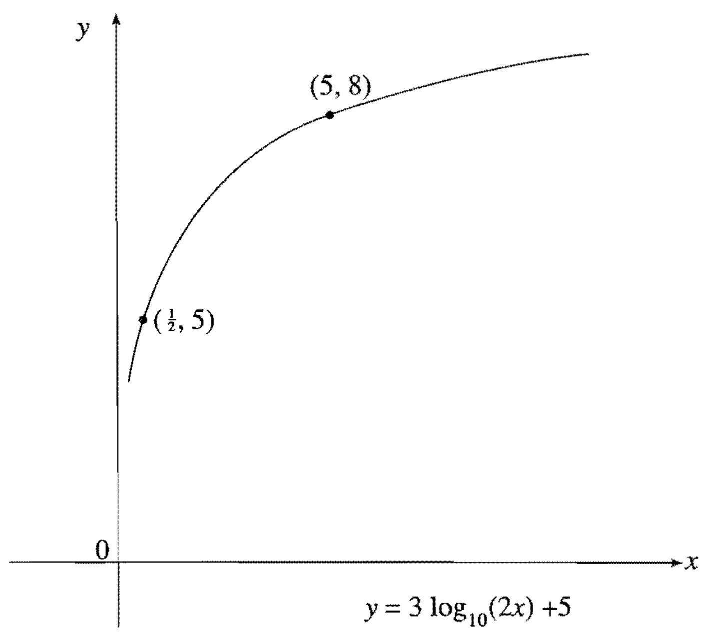
(ii)



(iii)



(iv)



Chapter 6

Exercises 6.1

1. $2x^2 + 2y^2 + x - 1 = 0$
2. $x^2 + y^2 = 25$
3. $|y + 1|,\ 2y = x^2 - 1$
4. $x^2 + y^2 = 5$
5. $y^2 = 4ax$
6. $y^2 - 8x^2 - 20ax - 8a^2 = 0$

Exercises 6.2

1. (a) $x^2 + y^2 - 2y - 8 = 0$ (b) $x^2 + y^2 + 2x - 4y = 0$
   (c) $x^2 + y^2 - 4x - 6y - 3 = 0$ (d) $x^2 + y^2 + 2x + 2y = 0$
   (e) $x^2 + y^2 - 8x - 2y + 12 = 0$
2. (a) $(-2, -1), 1$ (b) $(1, 2), 3$ (c) $\left(0, \frac{3}{2}\right), \frac{\sqrt{57}}{2}$
   (d) $(2, 0), 2$ (e) $\left(1, \frac{7}{8}\right), \frac{\sqrt{145}}{8}$ (f) $(0, 0), \frac{3}{2}$
3. $x^2 + y^2 - 4x + 2y + 1 = 0$
4. $x^2 + y^2 - 12x - 13y + 36 = 0$
5. $x^2 + y^2 - 5y + 5 = 0$
6. (a) $(-g, -f), \sqrt{g^2 + f^2 - c}$ (b) $\sqrt{g^2 + f^2 + 2g\alpha + 2f\beta + \alpha^2 + \beta^2}$

Exercises 6.3

1. (a) $y + x - 4 = 0$ (b) $2y + 3x - 5 = 0$ (c) $y + 4x - 11 = 0$
   (d) $4y + 9x - 5 = 0$
2. $\sqrt{26}$
3. $y - x + 1 = 0,\ y + x + 5 = 0$
4. $A(-5, 0),\ B\left(0, \frac{5}{2}\right),\ 6.25$
5. (a) $y + x = 0, y - x = 0$

Exercises 6.4

1. C is on the circle
2. $2\sqrt{17}$
3. $(3, 0)$
4. $\pm\frac{3}{4}, y = \frac{3}{4}x, y = -\frac{3}{4}x$
5. (a) $c = 2 - m$ (b) $c = \pm 2\sqrt{1 + m^2}$
   (c) $y = 2,\quad 3y + 4x - 10 = 0$
6. $4y - 3x + 10 = 0,\ 4y - 3x - 10 = 0$

Exercises 6.5

4. $\left(\frac{27}{25}, \frac{36}{25}\right)$
5. $(0, 0),\ 3y + x = 0$
6. 5

Chapter 7

Exercises 7.1

(i) not composite

(ii) composite, $g(x) = x^3 + 2x + 1$; $f(x) = \sqrt{x}$

(iii) composite, $g(x) = 5x + 7$, $f(x) = \tan x$

(iv) not composite

(v) composite, $g(x) = x^2 + 3$, $f(x) = x^{\frac{5}{3}}$

(vi) not composite

(vii) composite, $g(x) = x + 3$, $f(x) = x^2 + 5$

(viii) $g(x) = 6^x$, $f(x) = x + 7$ (ix) not composite

Exercises 7.2

(i) $2(2x - 3).2$ (ii) $2(3x^2 + 4).6x$ (iii) $2(x^3 + x)(3x^2 + 1)$

Exercises 7.3

(i) $3(x + 1)^2.1$ (ii) $3(2x - 1)^2.2$ (iii) $3(x^2 + 1)^2.2x$

Exercises 7.4

1. (i) $36(9x - 2)^3$ (ii) $-6x(3x^2 + 2)^{-2}$

(iii) $2(x^2 + 3x + 4)(2x + 3)$ (iv) $(2x+1)^{-\frac{1}{2}}$

(v) $3x^2(x^7 + 4x^3)^2(7x^4 + 12x^2)$ (vi) $-\dfrac{1}{(x+1)^2}$

(vii) $-5(x^2 - 4x + 2)^{-\frac{7}{2}}(x - 2)$ (viii) $-\dfrac{3}{2(3x+2)^{\frac{3}{2}}}$

(ix) $4\left(x + \dfrac{1}{x}\right)^3\left(1 - \dfrac{1}{x^2}\right)$ (x) $\dfrac{1}{2}\left(x^2 + \dfrac{1}{x}\right)^{-\frac{1}{2}}\left(2x - \dfrac{1}{x^2}\right)$

(xi) $\dfrac{1}{2}\left(3x + \dfrac{1}{x} + \dfrac{1}{x^2}\right)^{-\frac{1}{2}}\left(3 - \dfrac{1}{x^2} - \dfrac{2}{x^3}\right)$ (xii) $-\dfrac{1}{2}\left(7x - \dfrac{4}{x}\right)^{-\frac{3}{2}}\left(7 + \dfrac{4}{x^2}\right)$

Exercises 7.5

(i) $-4x^{-5}$ (ii) $12\left(x + \dfrac{1}{x}\right)^{11}\left(1 - \dfrac{1}{x^2}\right)$ (iii) $\dfrac{5}{2}\left(3x^2 + 5x - 61\right)^{\frac{3}{2}}(6x + 5)$

(iv) $-\dfrac{5}{2}\left(9x^4 - 7x^3 - \dfrac{3}{x^2}\right)^{-\frac{5}{2}}\left(36x^3 - 21x^2 + \dfrac{6}{x^3}\right)$

(v) $\dfrac{-\left(63x^8 - 18x^5 + 2\right)}{\left(7x^9 - 3x^6 + 2x + 1\right)^2}$ (vi) $\dfrac{-21x\left(x^5 - 1\right)}{\left(2x^7 - 7x^2 + 1\right)^{\frac{5}{2}}}$

(vii) $\dfrac{-\left(6x + 5 + \dfrac{3}{x^4}\right)}{2\left(3x^2 + 5x - \dfrac{1}{x^3}\right)^{\frac{3}{2}}}$ (viii) $-6\left(\sqrt{x} + \dfrac{2}{\sqrt{x}} + 3\right)^{-7}\left(\dfrac{1}{2\sqrt{x}} - \dfrac{1}{x^{\frac{3}{2}}}\right)$

Exercises 7.6

1. 0.99×2.7^x **2.** 1.001×2.72^x

Exercises 7.7

(i) $3e^{3x}$ (ii) $2xe^{x^2}$ (iii) $3x^2e^{x^3+2}$ (iv) $e^{x+\frac{1}{x}}\left(1 - \dfrac{1}{x^2}\right)$

(v) $-e^{-x}$ (vi) $-4e^{-4x}$ (vii) $\left(3x^2 - 1\right)e^{x^3-x+1}$

Exercises 7.8

1. (i) $\frac{1}{x}$ (ii) $\frac{6}{6x+5}$ (iii) $\frac{2x+1}{x^2+x}$ (iv) $-\frac{2}{x}$ (v) $\frac{18x+4}{9x^2+4x+3}$

(vi) $\frac{2x^3-1}{x(x^3+1)}$ (vii) $\frac{7x^6}{x^7+1}$ (viii) $-\frac{5}{2x}$ (ix) $\frac{2}{x+1}$ (x) $\frac{3(2x+1)}{x^2+x}$

(xi) 2 (xii) 1

2. (i) $2x$ (ii) $\frac{3}{x}(\ln x)^2$ (iii) $3x^2$ (iv) $\frac{3}{x}e^{3\ln x}$ which equals $3x^2$, in fact.

3. $\frac{3}{x}, \frac{4}{x}, \frac{7}{x}$ **6.** x

Exercises 7.9

(i) $\frac{1}{x}+e^x$ (ii) $\frac{2x}{x^2+1}+2x$ (iii) $3e^{3x}+4x^3$ (iv) $\frac{2x+1}{x^2+x}+3e^{3x-7}$

(v) $\frac{e^x+1}{e^x+x}$ (vi) $\frac{2xe^{x^2}+1}{e^{x^2}+x}$ (vii) $\frac{6x}{3x^2+2}+2(x-5)$ (viii) $\frac{e^x-\frac{1}{x^2}}{e^x+\frac{1}{x}+2}$

(ix) $\frac{e^x-e^{-x}}{e^x+e^{-x}}$ (x) $e^{3\ln x+x^2}\left(\frac{3}{x}+2x\right)\left(\text{or } x^3e^{x^2}\left(\frac{3}{x}+2x\right)\right)$

(xi) $4(e^x-x+2)^3(e^x-1)$

Exercises 7.10

(i) $\frac{2}{(x+1)^2}$ (ii) $\frac{\ln x-1}{(\ln x)^2}$ (iii) $\frac{(x+1)e^x}{(x+2)^2}$ (iv) $\frac{-30}{(5+3x)^2}$

(v) $\frac{-1}{(x+1)^2}$ (vi) $\frac{2e^x}{(e^x+1)^2}$ (vii) $\frac{\frac{1}{x}-\ln x}{e^x}$

(viii) $\frac{2(x^2+2x-3)}{(x^2+3)^2}=\frac{2(x-1)(x+3)}{(x^2+3)^2}$ (ix) $\frac{4}{(e^x+e^{-x})^2}$

Exercises 7.11

1. (i) $3x^2-6x+1-\frac{1}{x^2}$ (ii) $1+\ln x$ (iii) $\frac{8x^3}{(x^4+1)^2}$ (iv) $30x(x^2+1)^{14}$

(v) $\frac{-1}{2\sqrt{1-x}}$ (vi) $-4e^{-4x}$ (vii) $(\ln x)^2+2\ln x$ (viii) $\frac{-1}{2(x+1)^{\frac{3}{2}}}$

(ix) $e^x(1+\ln(e^x+1))$ (x) $\frac{e^{-x}-e^x}{(e^x+e^{-x})^2}$ (xi) $1-2x$

(xii) $(1-x)^9(1-11x)$ (xiii) $1-\frac{1}{x^2}$ (xiv) $1+\ln x+\frac{1}{x^2}(1-\ln x)$

(xv) $\frac{x^2(1-2\ln x)+1}{x(x^2+1)^2}$

2. $\frac{1}{2}$ **4.** $3e^2$ **5.** $\left(1,\frac{1}{2}\right)$ maximum ; $\left(-1,-\frac{1}{2}\right)$ minimum

6. (1, 0) maximum ; (3, 4) minimum **7.** (0, 0) minimum ; $(2, 4e^{-2})$ maximum

9. (i) $2^x \ln 2$ (ii) $3^x(1 + x\ln 3)$ (iii) $\dfrac{5^x(x\ln 5 - 1)}{x^2}$

(iv) $3^x\left[\dfrac{3}{3x+1} + \ln 3 \ln(3x+1)\right]$ (v) $3^x e^x(1 + \ln 3)$

Chapter 8
Exercises 8.2

1. (i) $3\cos x$ (ii) $-3\sin 3x$ (iii) $\dfrac{1}{2}\cos\dfrac{x}{2}$ (iv) $\dfrac{1}{4}\sec^2\left(\dfrac{x}{4}\right)$

(v) $\dfrac{3}{4}\sec\left(\dfrac{3x}{4}\right)\tan\left(\dfrac{3x}{4}\right)$ (vi) $-2\operatorname{cosec} 2x \cot 2x$

(vii) $3(\cos 3x - \sin 3x)$ (viii) $\sec x(\tan x + \sec x)$

(ix) $-\sin\left(\dfrac{x}{2}\right)$ (x) $\dfrac{1}{3x^{\frac{2}{3}}}\cos x - x^{\frac{1}{3}}\sin x$

(xi) $2x(1 - x)\sin x + (x^2 + 4x + 1)\cos x$

(xii) $-\dfrac{6}{x^4} + x(5\cos x - \sec^2 x) + 5\sin x - \tan x$

(xiii) $-6x\cos^2(x^2)\sin(x^2)$ (xiv) $\dfrac{\sec^2 x}{2\sqrt{\tan x}}$

(xv) $2x(\cos 2x - x\sin 2x)$ (xvi) $\sqrt{\sin x} + \dfrac{x\cos x}{2\sqrt{\sin x}}$

(xvii) 0 (xviii) $\dfrac{-(2x+3)\sin x - 2\cos x}{(2x+3)^2}$ (xix) $\dfrac{e^{3x}(3\cos x + \sin x)}{\cos^2 x}$

(xx) $\dfrac{-4x\sin 2x - \cos 2x}{2x^{\frac{3}{2}}}$ (xxi) $\dfrac{1+\cos^2 x}{2\cos^{\frac{3}{2}} x}$ (xxii) $-\tan x$

(xxiii) $-\operatorname{cosec} x$ (xxiv) $\dfrac{\sin x\cos x}{\sqrt{2 + \sin^2 x}}$

Exercises 8.3

1. (i) Minimum value $-\dfrac{\sqrt{3}}{2} - \dfrac{7\pi}{12} \approx -2.70$

Maximum value $\dfrac{\sqrt{3}}{2} - \dfrac{11\pi}{12} = -2.01$

(ii) Minimum value -2, maximum value 2

(iii) Minimum value ≈ -127.6, maximum value ≈ 5.51

(iv) Minimum value $-\dfrac{3}{4}\sqrt{3} \approx -1.299$, maximum value ≈ 1.299

(v) Minimum value 3, maximum value 11

2. $13^{\frac{3}{2}}$ or 46.87 approximately **4.** $\pm 54.7°$ approximately

5. Maximum value 1, minimum value $\dfrac{1}{\sqrt{2}} \approx 0.71$

Chapter 9

Exercises 9.1

1. (i) $\frac{x^7}{7}$ (ii) $\frac{3}{4}x^{4/3}$ (iii) $-\frac{4}{3x^3}$ (1v) $\frac{2}{3}x^{3/2}$

(v) $\frac{x^2}{2}-\frac{1}{x}$ (vi) $\frac{y^5}{5}+2y-\frac{1}{3y^3}$

Exercises 9.2

1. (i) $\frac{(x+1)^3}{3}$ (ii) $\frac{(2x-1)^4}{8}$ (iii) $\frac{(3x+7)^5}{15}$ (iv) $\frac{-(7x-6)^{-5}}{35}$

(v) $\frac{2(3x+1)^{\frac{3}{2}}}{9}$ (vi) $\frac{2(9x-8)^{\frac{1}{2}}}{9}$ (vii) $\frac{-1}{(2x+3)^{\frac{1}{2}}}$

(viii) $2\sqrt{1+x}$ (ix) $-\frac{(3-2x)^{\frac{5}{2}}}{5}-\frac{(3-2x)^{\frac{3}{2}}}{3}$ (x) $\frac{(lx+m)^{s+1}}{(s+1)l}$

2. (i) and (iii).

Exercises 9.3

(i) $2\ln|x|$ (ii) $\frac{1}{3}\ln|x|$ (iii) $\ln|x+1|$ (iv) $\frac{1}{9}\ln|9x+7|$

(v) $-\ln|1-x|$ (vi) $-\ln|3-x|+\frac{1}{2}\ln|3+2x|$

Exercises 9.4

1. (i) $\frac{1}{2}e^{2x+1}$ (ii) $-e^{-x+3}$ (iii) $-\frac{e^{-2x}}{2}$ (iv) $-\frac{e^{5-3x}}{3}$ (v) $-\frac{e^{-4x}}{4}$

(vi) $\frac{e^{3x}}{3}-\frac{e^{-3x}}{3}$ (vii) $\frac{2}{5}e^{\frac{5x}{2}}$ (viii) $\frac{2}{5}e^{5x}+3e^{-2x}$

2. (i), (ii), (iii), (vi)

Exercises 9.5

1. (i) $-\cos(x+2)$ (ii) $\frac{1}{5}\sin 5x$ (iii) $\frac{1}{5}\cos(9-5x)$

(iv) $\frac{1}{4}\sin(4x-7)+\frac{3}{2}\cos(2x+5)$ (v) $\frac{2}{7}\sin(7x+1)-\frac{5}{3}\cos 3x$

2. (i), (vi).

Exercises 9.6

(i) $-\frac{1}{x}$ (ii) $\ln|x|$ (iii) $\frac{4}{7}x^{\frac{7}{4}}$ (iv) $\frac{x^2}{2}+3x$ (v) $\frac{2}{3}(x+3)^{\frac{3}{2}}$

(vi) $\frac{x^3}{3}-\frac{3x^2}{4}$ (vii) $\frac{1}{10}\ln|10x-9|$ (viii) e^{x+3} (ix) $-\frac{e^{5-9x}}{9}$

(x) $\frac{(3x+2)^{11}}{33}$ (xi) $\frac{-1}{4(2x+9)^2}$ (xii) $2x+6\ln|x|-\frac{3}{2x^2}$

(xiii) $\frac{4}{5}x^{\frac{5}{2}} + \ln|x|$ (xiv) $\frac{x^4}{4} + \frac{3x^2}{2} + 3\ln|x| - \frac{1}{2x^2}$ (xv) $\frac{3}{2}x^{\frac{2}{3}}$

(xvi) $\frac{(a+bt)^3}{3b}$ (xvii) $-\sqrt{3-2y}$ (xviii) $y^2 - \frac{5y^4}{4}$

(xix) $\ln|2+3y|$ (xx) $-\frac{1}{x} + \frac{4}{3x^3}$ (xxi) $-\frac{2}{\sqrt{t}} + 2\sqrt{3+5t}$

(xxii) $\frac{1}{2(13-5w)^2}$ (xxiii) $-\frac{1}{5}\cos 5x$ (xiv) $-\frac{3}{2}\cos\left(2y - \frac{\pi}{4}\right)$

(xxv) $\frac{4}{3}\sin 3y + \frac{6}{7}\cos(7y+5)$ (xxvi) $\frac{7}{2}\cos(3-2x) - 2\sin(10-x)$

Exercises 9.7

1. $\frac{3}{28}$ **2.** 2 **3.** 8.45, correct to two decimal places

4. (a) 2.75 (b) 4.93, both correct to two decimal places

5. e – 1

Exercises 9.8

Answers are given correct to three decimal places.

1. 0.202 **2.** 1.019 **3.** 0.524, 3.143 **4.** 2.133 **5.** 3.373

Chapter 10

Exercises 10.1

1. (i) 1.314 (ii) 4.92 (iii) –5.61 (vi) 3.61
(v) 5.16 (vi) 1 (vii) 1

2. (i) $\ln a = x$ (ii) $\log_{10} b = y$ (iii) $\log_d c = z$
(iv) $\log_{10} 1 = 0$ (v) $\ln(7.389056) = 2$ (approximately)
(vi) $\log_{10} 204.1738 = 2.31$ (approximately)

Exercises 10.2

1. (i) $\frac{1}{x+1}$ (ii) $\frac{2x}{x^2+1}$ (iii) $\frac{1}{x}$ (iv) $\frac{2}{x}$

Exercises 10.3

1. (i) $-4\ln x$ (ii) $\ln x + \frac{3}{2}\ln y$ (iii) $4\ln x + \frac{3}{2}\ln y$ (iv) $\frac{1}{3}\ln x$
(v) $\frac{2}{3}\ln x + 4\ln y - 3\ln z$ (vi) $1 + \ln a$ (vii) $-2 - 2\ln b$
(viii) $\frac{1}{2}\log_{10} x - \frac{1}{2}\log_{10} y$ (ix) $2\log_{10} x + \frac{3}{2}\log_{10} y - \frac{1}{2}\log_{10} 2$
(x) $\log_{10} x + \frac{1}{2}\log_{10} y - \frac{1}{3}\log_{10} z$

2. (i) (c) (ii) (b) (iii) (c) (iv) (d)
(v) (b) (vi) (d)

3. (i) $\sqrt{3}$ (ii) $\frac{e+1}{e}$ (iii) e

4. $y=\sqrt{5}x^{-\frac{1}{2}}$ **5.** $y=\dfrac{(x+1)^2(x^2+1)\,e^2}{x^3}$

6. (i) $\dfrac{2}{x}$ (ii) $\dfrac{1}{x+1}+\dfrac{1}{x+2}$ (iii) $\dfrac{2}{2x+1}+\dfrac{4}{x+2}-\dfrac{3}{3x-5}$

(iv) $\dfrac{3}{x-2}+\dfrac{4}{2x+5}$ (v) $\dfrac{3}{x+1}+\dfrac{12}{3x-12}-\dfrac{4}{2x+1}$

Chapter 11
Exercises 11.1

1. –2, 2, 3 **2.** $1, \dfrac{1+\sqrt{5}}{2}, \dfrac{1-\sqrt{5}}{2}$ **3.** 6

4. $-5, \dfrac{1}{2}, 3$ **5.** –2

Exercises 11.2
1. 1.8 **2.** 3.7 **3.** –0.2

Exercises 11.3
1. -0.414 **2.** 1.3247 **3.** 1.0837

4. 0.567 **5.** 1.166 **6.** 3.1038

Exercises 11.4
1. $\dfrac{\ln 7}{\ln 3}$ **2.** $-\dfrac{\ln 3}{2\ln 3-\ln 2}=-\dfrac{\ln 3}{\ln\left(\frac{9}{2}\right)}$

3. $\dfrac{\ln 12}{\ln\left(\frac{4}{3}\right)}$ **4.** $\dfrac{\ln 6}{\ln 2}$ **5.** $\dfrac{\ln 3}{\ln 5}, \dfrac{\ln 5}{\ln 3}$

6. $3, \dfrac{3}{2}, -2;\ \dfrac{\ln 3}{\ln 5}, \dfrac{\ln\left(\frac{3}{2}\right)}{\ln 5}$ **7.** 4.505, 1.370

Chapter 12
Exercises 12.2
There are many possible counter-examples in addition to the following suggestions

1. $\theta=\dfrac{\pi}{4}$ **2.** $x=1, y=2$ **3.** $x=2, y=-3$

4. $x^2-2x+1=0$ **5.** $f(x)=x^4$ has a minimum at $x=0$

6. $p=2, n=3$

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