GCE AS/A LEVEL

WJEC GCE AS/A Level in MATHEMATICS

APPROVED BY QUALIFICATIONS WALES

GUIDANCE FOR TEACHING

Teaching from 2017

This Qualifications Wales regulated qualification is not available to centres in England.

CONTENTS

INTRODUCTION

The **WJEC AS and A Level Mathematics** qualifications, approved by Qualification Wales for first teaching from September 2017, are available to:

- All schools and colleges in Wales
- Schools and colleges in independent regions such as Northern Ireland, Isle of Man and the Channel Islands

The AS will be awarded for the first time in Summer 2018, using grades A – E; the A level will be awarded for the first time in Summer 2018, using grades A* – E.

The qualification provides a broad, coherent, satisfying and worthwhile course of study. It encourages learners to develop confidence in, and a positive attitude towards, mathematics and to recognise its importance in their own lives and to society.

The specification builds on the tradition and reputation WJEC has established for clear, reliable assessment supported by straightforward, accessible guidance and administration.

In addition to this guide, support is provided in the following ways:

- Specimen assessment materials
- Face-to-face CPD events
- Examiners' reports on each question paper
- Free access to past question papers and mark schemes via the secure website
- Direct access to the subject officer
- Free online resources
- Exam Results Analysis
- Online Examination Review

AIMS OF THE GUIDANCE FOR TEACHING

The principal aims of this Guidance for Teaching are to offer support to teachers in delivery of the new WJEC GCE AS/A Level Mathematics specification and to offer guidance on the requirements of the qualification and the assessment process.

The guide is **not intended as a comprehensive reference**, but as support for professional teachers to develop stimulating and exciting courses tailored to the needs and skills of their own students in their particular institutions.

The guide contains detailed clarification and guidance on the subject content for all units in the qualification.

The guide also contains a section on assessment objectives and how the different elements of these can be assessed in examination papers.

Written examination: 2 hours 30 minutes
25% of A level qualification (62.5% of AS qualification)
120 marks

Topics	Guidance
Proof	
Understand and use the structure of mathematical proof, proceeding from given assumptions through a series of logical steps to a conclusion; use methods of proof, including (a) proof by deduction, (b) proof by exhaustion, (c) disproof by counter example.	Proof by deduction to include the proofs of the laws of logarithms and differentiation from first principles for small positive powers of x. Learners will also be expected to generate their own proofs by applying the structure and techniques associated with standard mathematical proofs.

Topics	Guidance
Algebra and Functions	
Understand and use the laws of indices for all rational exponents. Use and manipulate surds, including rationalising the denominator.	Know the rules of indices: $a^p \times a^q = a^{p+q}$, $a^p \div a^q = a^{p-q}$, $a^p \times a^q = a^{pq}$, $\left(a^p\right)^q = a^{pq}$, $a^{-n} = \frac{1}{a^n}$, $\left(\sqrt[q]{a}\right)^p = a^{\frac{p}{q}}$ and $a^0 = 1$. To include rationalising fractions such as $\frac{2+3\sqrt{5}}{3-2\sqrt{5}}$ and $\frac{\sqrt{7}+\sqrt{5}}{\sqrt{7}-\sqrt{5}}$. Learners will be expected to know and use the results: $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$, $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$, $\left(\sqrt{a}\right)^2 = a$. Learners will be expected to give answers in their simplest form, eg. cancelling common factors in the numerator and denominator, $\frac{6+4\sqrt{5}}{2} = 3+2\sqrt{5}$

Topics	Guidance
Work with quadratic functions and their graphs. The discriminant of a quadratic function, including the conditions for real roots and repeated roots.	The nature of the roots of a quadratic equation. Learners will be expected to understand and use the following conditions: $b^2-4ac>0$ 2 real and distinct roots $b^2-4ac=0$ 2 equal real roots / 1 repeated real root $b^2-4ac<0$ no real roots
Completing the square.	Learners should be able to complete the square of a polynomial of the form ax^2+bx+c, where $a \neq 1$. To include finding the maximum or minimum value of a quadratic function, eg. $y=2(x-3)^2+4$ has a minimum at (3,4).
Solution of quadratic equations in a function of the unknown.	To include by factorisation, use of the formula and completing the square. Unknowns include powers of *x*, trigonometric, exponential or logarithmic functions, eg. $x^4-2x^2+1=0$
Solve simultaneous equations in two variables by elimination and by substitution, including one linear and one quadratic equation.	To include finding the points of intersection or the point of contact of a line and a curve. Eg. Solve $\begin{array}{l}2x=y-1\\x^2-3y+11=0\end{array}$ or $\begin{array}{l}y^2+xy=3\\2x+y=1\end{array}$ or $\begin{array}{l}x^2+4y^2=2\\2y+x+2=0\end{array}$

Topics	Guidance
Solve linear and quadratic inequalities in a single variable and interpret such inequalities graphically, including inequalities with brackets and fractions.	To include the solution of inequalities such as $1 - 2x < 4x + 7$, $\frac{x}{2} \geq 2(1-3x)$ and $x^2 - 6x + 8 \geq 0$.
Express solutions through the correct use of 'and' and 'or', or through set notation.	To include: $x < a$ or $x < b$ = $\{x : x < \mathrm{a}\} \cup \{x : x > b\}$ $x > c$ and $x < d$ = $\{x : x > c\} \cap \{x : x < d\}$
Represent linear and quadratic inequalities graphically.	To include, for example, $y > x + 1$ (a strict inequality) and $y \geq ax^2 + bx + c$ (a non-strict inequality).
Manipulate polynomials algebraically, including expanding brackets and collecting like terms, factorisation and simple algebraic division; use of the Factor Theorem.	The use of the Factor Theorem will be restricted to cubic polynomials and the solution of cubic equations. Learners may be required to factorise expressions such as $x^3 + 2x^2 - x - 2$ and $x^3 - 7x - 6$. Learners should know that if • $f(c) = 0$ then $(x - c)$ is a factor of $f(x)$ • $f\left(\frac{c}{a}\right) = 0$ then $(ax - c)$ is a factor of $f(x)$

Topics	Guidance
Understand and use graphs of functions; sketch curves defined by simple equations, including polynomials. $y=\frac{a}{x}$ and $y=\frac{a}{x^2}$, including their vertical and horizontal asymptotes. Interpret algebraic solutions of equations graphically. Use intersection points of graphs of curves to solve equations. Understand and use proportional relationships and their graphs.	The equations will be restricted to the form $y = f(x)$. Learners may be required to sketch simple cubic functions, including those with repeated roots, eg. $y=x(x-2)^2$ Eg. The solution of $x^2+4y^2=2$ and $x+2y=2$ is $x=1, y=0.5$ (ie. only one point of intersection), therefore, $x+2y=2$ must be a tangent to the curve $x^2+4y^2=2$. Use of the proportional symbol $\propto$ or equation with an unknown constant $y=kx$.
Understand the effect of simple transformations on the graph of $y = f(x)$ including sketching associated graphs: $y = af(x)$, $y = f(x) + a$, $y = f(x + a)$, $y = f(ax)$.	To include describing the transformation of a function from given graphs. Only single transformations will be considered.

Topics	Guidance
Coordinate geometry in the (x, y) plane	
Understand and use the equation of a straight line, including the forms $y = mx + c$, $y - y_1 = m(x - x_1)$ and $ax + by + c = 0$; gradient conditions for two straight lines to be parallel or perpendicular. Be able to use straight line models in a variety of contexts.	To include: • finding the gradient, equation, length and midpoint of a line joining two given points; • the condition for two lines to be parallel ($m = m'$) or perpendicular $\left(m' = -\frac{1}{m}\right)$ • finding the equations of lines which are parallel or perpendicular to a given line; • finding the point of intersection of two lines. Eg. velocity against time for constant acceleration.
Understand and use the coordinate geometry of the circle using the equation of a circle in the form $(x - a)^2 + (y - b)^2 = r^2$; completing the square to find the centre and radius of a circle. Use of the following circle properties: (i) the angle in a semicircle is a right angle; (ii) the perpendicular from the centre to a chord bisects the chord; (iii) the radius of a circle at a given point on its circumference is perpendicular to the tangent to the circle at that point.	To also be familiar with the equation of a circle in the form $x^2 + y^2 + 2gx + 2fy + c = 0$. To include: • finding the equations of tangents, • the condition for two circles to touch internally or externally, • finding the points of intersection or the point of contact of a line and a circle.

Topics	Guidance
Sequences and Series - The Binomial Theorem	
Understand and use the binomial expansion of $(a + bx)^n$ for positive integer n . The notations $n!$, $\binom{n}{r}$ and $n\mathrm{C}r$. Link to binomial probabilities.	Eg. Find the coefficient of the x^3 term in the expansion of $(3-2x)^5$ To include use of Pascal's triangle.

Topics	Guidance
Trigonometry	
Understand and use the definitions of sine, cosine and tangent for all arguments.	Use of the exact values of the sine, cosine and tangent of 30°, 45° and 60°.
Understand and use the sine and cosine rules, and the area of a triangle in the form ½ $ab\sin C$.	To include the use of the sine rule in the ambiguous case. Sine rule: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ Cosine rule: $a^2 = b^2 + c^2 - 2bc\cos A$
Understand and use the sine, cosine and tangent functions. Understand and use their graphs, symmetries and periodicity.	Use of graphs to understand • $\cos\theta = \cos(360^\circ - \theta)$ and $\cos(-\theta) = \cos\theta$ • $\sin\theta = \sin(180^\circ - \theta)$ and $\sin(-\theta) = -\sin\theta$ • $\tan\theta = \tan(180^\circ + \theta)$ and $\tan(-\theta) = -\tan\theta$
Understand and use $\tan\theta = \frac{\sin\theta}{\cos\theta}$. Understand and use $\cos^2\theta + \sin^2\theta = 1$.	These identities may be used to solve trigonometric equations or prove trigonometric identities.
Solve simple trigonometric equations in a given interval, including quadratic equations in sin, cos and tan, and equations involving multiples of the unknown angle.	To include the solution of equations such as • $3\sin\theta = 1$, for $0^\circ \le \theta \le 360^\circ$ • $\tan\theta = \frac{\sqrt{3}}{2}$, for $-180^\circ \le \theta \le 180^\circ$ • $3\cos 2\theta = -1$, for $0^\circ \le \theta \le 180^\circ$ • $2\cos^2\theta + \sin\theta - 1 = 0$, for $0^\circ \le \theta < 360^\circ$ • $3\sin^2\theta + 5\cos\theta - 5 = 0$, for $0^\circ < \theta \le 360^\circ$

Topics	Guidance
Exponentials and logarithms	
Know and use the function a^x and its graph, where a is positive. Know and use the function e^x and its graph.	Know that all graphs of the form $y = a^x$, where a is positive, pass through the point (0,1).
Know that the gradient of e^{kx} is equal to $k\mathrm{e}^{kx}$ and hence understand why the exponential model is suitable in many applications.	Realise that when the rate of change is proportional to the y value, ie. $\frac{\mathrm{d}y}{\mathrm{d}x} \propto y$, an exponential model should be used.
Know and use the definition of $\log_a x$ as the inverse of a^x, where a is positive and $x \geq 0$. Know and use the function $\ln x$ and its graph. Know and use $\ln x$ as the inverse function of e^x.	Learners should be able to convert from index to logarithmic form and vice versa, ie. $p = q^r \Leftrightarrow r = \log_q p$. Know that $\log_c c = 1$ and $\log_c 1 = 0$. For example: • Given $y = 4$ when $x = 3$, find the value of k in the equation $y = e^{kx}$ • Solve $\mathrm{e}^{2x-1} = 3$ • Solve $\ln(4x+3) = 2$
Understand and use the laws of logarithms. $\log_a x + \log_a y = \log_a (xy)$ $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$ $k\log_a x = \log_a (x^k)$ (including, for example $k = -1$, $k = -½$)	To include the proof of the laws of logarithms. Use of the laws of logarithms to solve equations and to simplify expressions involving logarithms. e.g. Simplify $\log_2 36 - 2\log_2 15 + \log_2 100 + 1$. Change of base will not be required.

Topics	Guidance
Solve equations in the form $a^x = b$.	The use of a calculator to solve equations such as (i) $3^x = 2$, (ii) $25^x - 4 \times 5^x + 3 = 0$. (iii) $4^{2x+1} = 5^x$
Use logarithmic graphs to estimate parameters in relationships of the form $y = ax^n$ and $y = kb^x$, given data for x and y.	Link to laws of logarithms. Understand that on a graph of $\log y$ against $\log x$, the gradient is n and the intercept is $\log a$, and that on a graph of $\log y$ against x, the gradient is $\log b$ and the intercept is $\log k$.
Understand and use exponential growth and decay; use in modelling (examples may include the use of e in continuous compound interest, radioactive decay, drug concentration decay, exponential growth as model for population growth.) Consideration of limitations and refinements of exponential models.	The formal differentiation and integration of formulae involving e^x and/or a^x will not be required. Be familiar with the meaning of the constants used in the model and to find values for them given appropriate conditions. Explore the behaviour of the variables for large values of t. Comment on the appropriateness of predicted values using the model.

Topics	Guidance
Differentiation	
Understand and use the derivative of $f(x)$ as the gradient of the tangent to the graph of $y = f(x)$ at a general point (x, y); the gradient of the tangent as a limit; interpretation as a rate of change; sketching the gradient function for a given curve; second order derivatives.	Learners should know that $\frac{\mathrm{d}y}{\mathrm{d}x}$ denotes the rate of change of y with respect to x. The notation $\frac{\mathrm{d}y}{\mathrm{d}x}$ or $f'(x)$ may be used for the first derivative and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ or $f''(x)$ for the second derivative. Knowledge of the chain rule is not required here.
Differentiation from first principles for small positive integer powers of x.	Up to and including power of 3. To include polynomials up to and including a maximum degree of 3. Eg. Differentiate from first principles $y = x^3 - 2$
Understand and use the second derivative as the rate of change of gradient.	Learners should understand that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$. Be able to use the second derivative to distinguish between maximum and minimum points, ie. $f''(x) > 0$ implies a minimum and $f''(x) < 0$ implies a maximum for points where $f'(x) = 0$.
Differentiate x^n for rational n, and related constant multiples, sums and differences.	To include polynomials. May require the removal of brackets eg $(3x^2 - 1)(x + 7)$.

Topics	Guidance
Apply differentiation to find gradients, tangents and normals, maxima and minima, and stationary points. Identify where functions are increasing or decreasing.	To include finding the equations of tangents and normals. The use of maxima and minima in simple optimisation problems. To include simple curve sketching. Use of the conditions $\left\|f'(x)\right\| > 0$ for an increasing function and $\left\|f'(x)\right\| < 0$ for an increasing function.
Integration	
Know and use the Fundamental Theorem of Calculus.	Integration as the reverse process of differentiation. The constant of integration is required for indefinite integrals.
Integrate x^n (excluding n = – 1) and related sums, differences and constant multiples.	To include polynomials and expressions such as $2x^2 - \sqrt{x}$, $\dfrac{x+1}{\sqrt{x}}$.
Evaluate definite integrals. Use a definite integral to find the area under a curve.	To include finding the area of a region between a straight line and a curve. The area may include a region below the x-axis.

Topics	Guidance
Vectors	
Use vectors in two dimensions.	To include the use of the unit vectors, $\mathbf{i}$ and $\mathbf{j}$. Learners should be familiar with column vectors and the notation $\overrightarrow{CD}$ and $\mathbf{c}$.
Add vectors diagrammatically and perform the algebraic operations of vector addition and multiplication by scalars, and understand their geometrical interpretations.	To include the condition for two vectors to be parallel.
Understand and use position vectors; calculate the distance between points represented by position vectors. Use vectors to solve problems in pure mathematics.	Use of $\mathbf{AB} = \mathbf{b} - \mathbf{a}$ or $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$. To include the use of position vectors given in terms of unit vectors. To include the use and derivation of the position vector of a point dividing a line in a given ratio.

Written examination: 1 hour 45 minutes
15% of A level qualification (37.5% of AS qualification)
75 marks

The paper will comprise two sections:

Section A: Statistics (40 marks)

Section B: Mechanics (35 marks)

Topics	Guidance
STATISTICS	
2.2.1 Statistical Sampling	
Understand and use the terms 'population' and 'sample'. Use samples to make informal inferences about the population. Understand and use sampling techniques, including simple random sampling, systematic sampling and opportunity sampling.	
Select or critique sampling techniques in the context of solving a statistical problem, including understanding that different samples can lead to different conclusions about the population.	

Topics	Guidance
2.2.2 Data presentation and interpretation	
Interpret diagrams for single-variable data, including understanding that area in a histogram represents frequency. Connect to probability distributions.	Learners should be familiar with box and whisker diagrams and cumulative frequency diagrams. Qualitative assessment of skewness is expected and the use of the terms symmetric, positive skew or negative skew
Interpret scatter diagrams and regression lines for bivariate data, including recognition of scatter diagrams which include distinct sections of the population. (Calculations of coefficients of regression lines are excluded.) Understand informal interpretation of correlation. Understand that correlation does not imply causation.	Equations of regression lines may be given in a question and learners asked to make predictions using it. Use of the terms positive, negative, zero, strong and weak is expected.
Interpret measures of central tendency and variation, extending to standard deviation. Be able to calculate standard deviation, including from summary statistics.	Measures of central tendency: mean, median, mode. Measures of central variation: variance, standard deviation, range, interquartile range.
Recognise and interpret possible outliers in data sets and statistical diagrams. Select or critique data presentation techniques in the context of a statistical problem. Be able to clean data, including dealing with missing data, errors and outliers.	Use of $Q_1 - 1.5 \times \text{IQR}$ and $Q_3 + 1.5 \times \text{IQR}$ to identify outliers.

Topics	Guidance
2.2.3 Probability	
Understand and use mutually exclusive and independent events when calculating probabilities. Link to discrete and continuous distributions.	To include the multiplication law for independent events: $\mathrm{P}(A \cap B) = \mathrm{P}(A)\,\mathrm{P}(B)$.
Use Venn diagrams to calculate probabilities.	Use of set notation and associated language is expected. To include the generalised addition law: $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$. Conditional probability will not be assessed in this unit.
2.2.4 Statistical distributions	
Understand and use simple, discrete probability distributions. Understand and use, • the binomial distribution, as a model • the Poisson distribution, as a model • the discrete uniform distribution, as a model (Calculation of mean and variance of discrete random variables is excluded.)	To include using distributions to model real world situations and to comment on their appropriateness.

Topics	Guidance
Calculate probabilities using • the binomial distribution. • the Poisson distribution. • the discrete uniform distribution.	Use of the binomial formula and tables / calculator. Use of the Poisson formula and tables / calculator Use of the formula for the discrete uniform distribution.
Select an appropriate probability distribution for a context, with appropriate reasoning, including recognising when the binomial, Poisson or discrete uniform model may not be appropriate.	
2.2.5 Statistical hypothesis testing	
Understand and apply the language of statistical hypothesis testing, developed through a binomial model: null hypothesis, alternative hypothesis, significance level, test statistic, 1-tail test, 2-tail test, critical value, critical region, acceptance region, *p*-value.	The *p*-value is the probability that the observed result or a more extreme one will occur under the null hypothesis H_0. For uniformity, interpretations of a *p*-value should be along the following lines: $p < 0{\cdot}01$; there is very strong evidence for rejecting H_0. $0{\cdot}01 \le p \le 0{\cdot}05$; there is strong evidence for rejecting H_0. $p > 0{\cdot}05$; there is insufficient evidence for rejecting H_0.
Conduct a statistical hypothesis test for the proportion in the binomial distribution and interpret the results in context. Understand that a sample is being used to make an inference about the population and appreciate that the significance level is the probability of incorrectly rejecting the null hypothesis.	
Interpret and calculate Type I and Type II errors, and know their practical meaning.	

Topics	Guidance
MECHANICS	
2.2.6 Quantities and units in mechanics	
Understand and use fundamental quantities and units in the S.I. system; length, time and mass.	
Understand and use derived quantities and units: velocity, acceleration, force, weight.	
2.2.7 Kinematics	
Understand and use the language of kinematics: position, displacement, distance travelled, velocity, speed, acceleration.	
Understand, use and interpret graphs in kinematics for motion in a straight line: displacement against time and interpretation of the gradient; velocity against time and interpretation of the gradient and the area under the graph	Learners may be expected to sketch displacement-time and velocity-time graphs.
Understand, use and derive the formulae for constant acceleration for motion in a straight line.	To include vertical motion under gravity. Gravitational acceleration, g. The inverse square law for gravitation is not required and g may be assumed to be constant, but learners should be aware that g is not a universal constant but depends on location. The value $9{\cdot}8$ ms^{-2} can be used for the acceleration due to gravity, unless explicitly stated otherwise.

Topics	Guidance
Use calculus in kinematics for motion in a straight line.	To include the use of $v = \frac{\mathrm{d}r}{\mathrm{d}t}$, $a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 r}{\mathrm{d}t^2}$, $r = \int v\,\mathrm{d}t$, $v = \int a\,\mathrm{d}t$, where v, a and r are given in terms of t.
2.2.8 Forces and Newton's laws	
Understand the concept of a force. Understand and use Newton's first law.	
Understand and use Newton's second law for motion in a straight line (restricted to forces in two perpendicular directions or simple cases of forces given as 2-D vectors).	Use of **F** = *ma*
Understand and use weight and motion in a straight line under gravity; gravitational acceleration, g, and its value in S.I. units to varying degrees of accuracy. (The inverse square law for gravitation is not required and g may be assumed to be constant, but learners should be aware that g is not a universal constant but depends on location.)	Forces will be constant and will include weight, normal reaction, tension and thrust. To include problems involving lifts. The value 9·8 ms^{-2} can be used for the acceleration due to gravity, unless explicitly stated otherwise.
Understand and use Newton's third law. Equilibrium of forces on a particle and motion in a straight line (restricted to forces in two perpendicular directions or simple cases of forces given as 2-D vectors) Applications to problems involving smooth pulleys and connected particles.	Problems involving particles connected by strings passing over smooth, fixed pulleys or pegs; one particle will be freely hanging and the other particle may be (i) freely hanging, (ii) on a smooth, horizontal plane.

Topics	Guidance
2.2.9 Vectors	
Calculate the magnitude and direction of a vector and convert between component form and magnitude/direction form.	
Use vectors to solve problems in context, including forces.	Does not include kinematics problems.

Written examination : 2 hours 30 minutes
35% of A level qualification
120 marks

Topics	Guidance
2.3.1 Proof	
Proof by contradiction (including proof of the irrationality of $\sqrt{2}$ and the infinity of primes, and application to unfamiliar proofs).	
2.3.2 Algebra and Functions	
Simplify rational expressions, including by factorising and cancelling and by algebraic division (by linear expressions only).	Eg. $\dfrac{x^2-4}{x^2+x-2}$, $\dfrac{x^2+2x+1}{3x^2+12x+9}$
Sketch curves defined by the modulus of a linear function.	Be able sketch graphs of the form $y=\lvert ax+b\rvert$. To include solving equations involving the modulus function, eg. $\lvert 2x+3\rvert = x$

Topics	Guidance
Understand and use composite functions; inverse functions and their graphs.	Understand and use the definition of a function. Understand and use the domain and range of functions. In the case of a function defined by a formula (with unspecified domain) the domain is taken to be the largest set such that the formula gives a unique image for each element of the set. Graph of $y = f'(x)$ is the image of the graph of $y = f(x)$ after reflection in the line $y = x$. The notation $f^{-1}(x)$ will be used for the inverse function and *fg* will be used for composition to mean $f(g(x))$. Know the condition for the inverse function to exist and be able to find the inverse function algebraically. Given the function $y = f(x)$, learners should be able to sketch the inverse function of $y = f(x)$ by reflection in the line $y = x$.
Understand the effect of combinations of transformations on the graph of $y = f(x)$, as represented by $y = af(x)$, $y = f(x) + a$, $y = f(x + a)$ and $y = f(ax)$.	Learners should be able to apply combinations of these transformations to functions in the specification, eg e^x, $\lvert x \rvert$, $\cos x$ etc and sketch the resulting graphs.
Decompose rational functions into partial fractions (denominators not more complicated than squared linear terms and with no more than 3 terms, numerators constant or linear).	With denominators of the form $(ax + b)(cx + d)$, $(ax + b)(cx + d)(ex + f)$ and $(ax + b)(cx + d)^2$. Learners will **not** be expected to sketch the graphs of rational functions.
Use of functions in modelling, including consideration of limitations and refinements of the models.	Eg modelling the behaviour of tides with trigonometric functions, exponential growth and decay, modelling volume and pressure of gases.

Topics	Guidance
2.3.3 Coordinate geometry in the (x, y) plane	
Understand and use the parametric equations of curves and conversion between Cartesian and parametric forms.	To include finding the equations of tangents and normals to curves defined parametrically or implicitly. Eg. $x = \cos t$, $y = \sin t$ describes a circle with centre (0,0) and radius 4. Knowledge of the properties of curves other than the circle will **not** be expected.
Use parametric equations in modelling in a variety of contexts.	
2.3.4 Sequences and Series	
Understand and use the binomial expansion of $(a+bx)^n$, for any rational n, including its use for approximation. Be aware that the expansion is valid for $\left\|\frac{bx}{a}\right\| < 1$ (proof not required).	To include the expansion, in ascending powers of x, of expressions such as $(2-x)^{\frac{1}{2}}$ and $\frac{(4-x)^{\frac{3}{2}}}{(1+2x)}$ and be able to state the range of validity.
Work with sequences, including those given by a formula for the nth term and those generated by a simple relation of the form $x_{n+1} = f(x_n)$. Increasing sequences, decreasing sequences, periodic sequences.	To include being able to generate the terms of a sequence. Eg. $u_n = (-1)^n$ for $n > 0$ is a periodic function of order 2.
Understand and use sigma notation for sums of series.	Know the difference between a sequence and a series.

Topics	Guidance
Understand and work with arithmetic sequences and series, including the formulae for the *n*th term and the sum to *n* terms.	Use of $u_n = a + (n-1)d$. Use and proof of $S_n = \frac{n}{2}[2a + (n-1)d]$ and $S_n = \frac{n}{2}[a + l]$.
Understand and work with geometric sequences and series, including the formulae for the *n*th term and the sum of a finite geometric series. The sum to infinity of a convergent geometric series, including the use of $\lvert r \rvert < 1$; modulus notation.	Use of $u_n = ar^{n-1}$. Use and proof of $S_n = \frac{a(1-r^n)}{1-r}$. Use of $S_\infty = \frac{a}{1-r}$ for $\lvert r \rvert < 1$.
Use sequences and series in modelling.	Eg compound interest, loans.
2.3.5 Trigonometry	
Work with radian measure, including use for arc length, area of sector and area of segment.	Understand the relationship between degrees and radians. Use of degrees or radians in problems involving trig functions. Learners should know and be able to use the formulae $s = r\theta$ for arc length and $A = \frac{1}{2}r^2\theta$ for the area of a sector.
Understand and use the standard small angle approximations of sine, cosine and tangent. $\sin\theta \approx \theta$, $\cos\theta \approx 1 - \frac{\theta^2}{2}$ and $\tan\theta \approx \theta$, where θ is in radians.	Eg show $\frac{\cos^2 x - 1}{x \sin 2x} \approx -\frac{1}{2}$ for when x is small. Application to differentiation of $\sin x$ and $\cos x$ from first principles.

Topics	Guidance
Know and use exact values of sin and cos for $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi$ and multiples thereof, and exact values of tan for $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \pi$ and multiples thereof.	Eg Find the exact value of 165°.
Understand and use the definitions of sec, cosec, cot, $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$. Understand the relationships of all of these to sin, cos and tan and understand their graphs, ranges and domains.	For example, understand that $y = \sec x$ means $\frac{1}{\cos x}$ and $y = \cos^{-1} x$ means $\cos y = x$.
Understand and use $\sec^2\theta \equiv 1 + \tan^2\theta$ and $\operatorname{cosec}^2\theta \equiv 1 + \cot^2\theta$.	The solution of trigonometric equations such as $\sec^2\theta + 5 = 5\tan\theta$. May be used to prove trigonometric identities.
Understand and use double angle formulae. Use of formulae for $\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$. Understand geometric proofs of these formulae.	Use of these formulae to solve equations in a given range, e.g. $\sin 2\theta = \sin\theta$, Applications to integration, e.g. $\int \cos^2 x\, dx$.
Understand and use expressions for $a\cos\theta + b\sin\theta$ in the equivalent forms of $r\cos(\theta \pm \alpha)$ or $r\sin(\theta \pm \alpha)$.	Use of these to solve equations in a given range, e.g. $3\cos\theta + \sin\theta = 2$. Application to finding greatest and least values, e.g. the least value of $\frac{1}{3\cos\theta + 4\sin\theta + 10}$.
Construct proofs involving trigonometric functions and identities.	Eg. Prove that $\frac{\sin(A-B)}{\cos A \cos B} \equiv \tan A - \tan B$.

Topics	Guidance
2.3.6 Differentiation	
Differentiation from first principles for $\sin x$ and $\cos x$.	Use of compound angle formulae and small angle approximations, ie. for $\cos x$, use of $$\frac{\cos(x+\delta x)-\cos x}{\delta x}=\frac{\cos x\cos\delta x-\sin x\sin\delta x-\cos x}{\delta x}$$ and for $\sin x$, use of $$\frac{\sin(x+\delta x)-\sin x}{\delta x}=\frac{\sin x\cos\delta x-\cos x\sin\delta x-\sin x}{\delta x}$$ and let $\delta x \to 0$. Students may use δx or h.
Understand and use the second derivative as the rate of change of gradient; connection to convex and concave sections of curves, and points of inflection.	Points of inflection to include stationary and non-stationary points. Use of the condition that: • if $f'(x) > 0$ in an interval, the function is convex in that interval; • if $f'(x) < 0$ in an interval, the function is concave in that interval; • at a point of inflection $f''(x)$ changes sign.
Differentiate e^{kx}, a^{kx}, $\sin kx$, $\cos kx$, $\tan kx$, and related sums, differences and constant multiples. Understand and use the derivative of $\ln x$.	Use of the result $\frac{\mathrm{d}}{\mathrm{d}x}\left(a^{kx}\right)=ka^{kx}\ln a$.

Topics	Guidance
Apply differentiation to find points of inflection.	Learners should know that if x is a point of inflection then $f''(x)=0$ and the second derivative changes sign either side of x. In addition, • if $f'(x)=0$, then the point of inflection is a stationary point; • if $f'(x)\neq 0$, then the point of inflection is a non-stationary point.
Differentiate using the product rule, the quotient rule and the chain rule, including problems involving connected rates of change and inverse functions.	To include the use of $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)}$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\times\frac{\mathrm{d}t}{\mathrm{d}x}$. To include differentiation of $\sec x$, $\operatorname{cosec} x$, $\cot x$, $\cos^{-1} x$, $\sin^{-1} x$, $\tan^{-1} x$.
Differentiate simple functions and relations defined implicitly or parametrically, for first derivative only.	To include finding equations of tangents and normal to curves given parametrically or implicitly. Eg. Find the equation of the normal to the curve $x^3-2xy^2+y^3=5$ at the point (2, 1). Eg. The curve *C* has parametric equations $x=2t$, $y=5t^3$. Show that the equation of the tangent to *C* at the point *P* is $2y=15p^2x-20p^3$.
Construct simple differential equations in pure mathematics.	Only knowledge of first order differential equations will be required.

Topics	Guidance
2.3.7 Integration	
Integrate e^{kx} , $\frac{1}{x}$, $\sin kx$, $\cos kx$ and related sums, differences and constant multiples.	Use of the results: 1) if $\int \mathrm{f}(x)dx = \mathrm{F}(x) + k$ then $\int \mathrm{f}(ax+b)dx = \frac{1}{a}\mathrm{F}(ax+b)+c.$ 2) $\int f'(g(x))g'(x)dx = f(g(x))+c$ To include the use of trigonometric identities such as double angle formulae to integrate functions such as $\cos^2 x$ and $\sin^2 2x$.
Use a definite integral to find the area between two curves.	
Understand and use integration as the limit of a sum.	$\int_a^b f(x)\mathrm{d}x = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} f(x)\delta x$
Carry out simple cases of integration by substitution and integration by parts. Understand these methods as the reverse processes of the chain rule and the product rule respectively.	Use of $\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$. Learners will be expected to integrate functions such as $(4-3x)^5$, $\frac{3}{(2x-1)^4}$, $x\cos x$.
Integration by substitution includes finding a suitable substitution and is limited to cases where one substitution will lead to a function which can be integrated.	Eg. $\int x\sqrt{1+x^2}\,\mathrm{d}x$, $\int \frac{x^3}{1+x^4}\,\mathrm{d}x$, $\int \frac{\cos x}{\sin^3 x}\,\mathrm{d}x$
Integration by parts includes more than one application of the method but excludes reduction formulae.	Eg. $\int x^2\mathrm{e}^x\mathrm{d}x$

Topics	Guidance
Integrate using partial fractions that are linear in the denominator.	Eg. $\int \frac{5}{2+3x} \mathrm{d}x$
Evaluate the analytical solution of simple first order differential equations with separable variables, including finding particular solutions. (Separation of variables may require factorisation involving a common factor.)	Questions will be set in pure mathematics only.
2.3.8 Numerical Methods	
Locate roots of $f(x) = 0$ by considering changes in sign of $f(x)$ in an interval of x in which $f(x)$ is sufficiently well-behaved. Understand how change of sign methods can fail.	Eg if the interval is too large so as to include an even number of roots, or if the function is discontinuous or has an asymptote.
Solve equations approximately using simple iterative methods; be able to draw associated cobweb and staircase diagrams. Solve equations using the Newton-Raphson method and other recurrence relations of the form $x_{n+1} = g(x_n)$. Understand how such methods can fail.	The iterative formula will be given. Consideration of the conditions for convergence will **not** be required. Understand convergence in geometrical terms by drawing cobweb and staircase diagrams. Newton-Raphson: If $x = a$ is an approximate solution of the equation $f(x) = 0$ then a better approximation is $x = b$ where $b = a - \frac{f(a)}{f'(a)}$. Method fails if the gradient is too small.

Topics	Guidance
Understand and use numerical integration of functions, including the use of the trapezium rule and estimating the approximate area under a curve and limits that it must lie between.	Learners will be expected to use the trapezium rule to estimate the area under a curve and to determine whether it gives an overestimate or an underestimate of the area under a curve. May require the use of a sketched graph. Simpson's rule is excluded.
Use numerical methods to solve problems in context.	To solve problems in context which lead to equations that cannot be solved analytically.

Written examination: 1 hour 45 minutes
25% of A level qualification
80 marks

The paper will comprise two sections:

Section A: Statistics (40 marks)

Section B: Differential Equations and Mechanics (40 marks)

Topics	Guidance
STATISTICS	
2.4.1 Probability	
Understand and use conditional probability, including the use of tree diagrams, Venn diagrams and two-way tables.	
Understand and use the conditional probability formula: $P(A \cap B) = P(A)\,P(B \mid A) = P(B)\,P(A \mid B)$.	
Modelling with probability, including critiquing assumptions made and the likely effect of more realistic assumptions.	

Topics	Guidance
2.4.2 Statistical distributions	
Understand and use the continuous uniform distribution and Normal distributions as models. Find probabilities using the Normal distribution. Link to histograms, mean, standard deviation, points of inflection and the binomial distribution.	Use of calculator / tables to find probabilities. Linear interpolation in tables will **not** be required.
Select an appropriate probability distribution for a context, with appropriate reasoning, including recognising when the continuous uniform or Normal model may not be appropriate.	The distributions from which the selection can be made are: Discrete: binomial, Poisson, uniform Continuous: Normal, uniform
2.4.3 Statistical hypothesis testing	
Understand and apply statistical hypothesis testing to correlation coefficients as measures of how close data points lie to a straight line and be able to interpret a given correlation coefficient using a given *p*-value or critical value. (The calculation of correlation coefficients is excluded.)	Learners will be expected to state hypotheses in terms of ρ, where ρ represents the population correlation coefficient.
Conduct a statistical hypothesis test for the mean of a Normal distribution with known, given or assumed variance, and interpret the results in context.	Learners should know and be able to use the result that if $X \sim \mathrm{N}(\mu, \sigma^2)$ then $\overline{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$ (The proof is excluded.)

Topics	Guidance
DIFFERENTIAL EQUATIONS AND MECHANICS	
2.4.4 Trigonometry	
Use trigonometric functions to solve problems in context, including problems involving vectors, kinematics and forces.	Contexts may include, for example, wave motion as well as problems in vector form which involve resolving directions and quantities in mechanics.
2.4.5 Differentiation	
Construct simple differential equations in context (contexts may include kinematics, population growth and modelling the relationship between price and demand).	To include contexts involving exponential growth and decay.
2.4.6 Integration	
Evaluate the analytical solution of simple first order differential equations with separable variables, including finding particular solutions.	Questions will be set in context. Separation of variables may require factorisation involving a common factor.
Interpret the solution of a differential equation in the context of solving a problem, including identifying limitations of the solution; includes links to kinematics.	

Topics	Guidance
2.4.7 Quantities and units in mechanics	
Understand and use derived quantities and units for moments.	
2.4.8 Kinematics	
Extend, use and derive the formulae for constant acceleration for motion in a straight line to 2 dimensions using vectors.	$\mathbf{v} = \mathbf{u} + \mathbf{a}t$ etc.
Extend the use of calculus in kinematics for motion in a straight line to 2 dimensions using vectors.	To include the use of $\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$, $\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}$, $\mathbf{r} = \int \mathbf{v}\,\mathrm{d}t$, $\mathbf{v} = \int \mathbf{a}\,\mathrm{d}t$, where $\mathbf{v}$, $\mathbf{a}$ and $\mathbf{r}$ are given in terms of t.
Model motion under gravity in a vertical plane using vectors; projectiles.	To include finding the speed and direction of motion of the projectile at any point on its path. The maximum horizontal range of a projectile for a given speed of projection. In examination questions, learners may be expected to derive the general form of the formulae for the range, the time of flight, the greatest height or the equation of path. In questions where derivation of formulae has not been requested, the quoting of these formulae will **not** gain full credit. Questions will **not** involve resistive forces.

Topics	Guidance
2.4.9 Forces and Newton's laws	
Extend Newton's second law to situations where forces need to be resolved (restricted to two dimensions).	Eg motion on an inclined plane.
Resolve forces in two dimensions. Understand and use the equilibrium of a particle under coplanar forces.	
Understand and use addition of forces; resultant forces; dynamics for motion in a plane.	Forces may need to be resolved into 2 components.
Understand and use the $\mathrm{F} \leq \mu \mathrm{R}$ model for friction. The coefficient of friction. The motion of a body on a rough surface. Limiting friction and statics.	Forces will be constant and will include weight, friction, normal reaction, tension and thrust. To include motion on an inclined plane. The motion of particles connected by strings passing over smooth, fixed pulleys or pegs; one particle will be freely hanging and the other particle may be on an inclined plane.
2.4.10 Moments	
Understand and use moments in simple static contexts.	To include parallel forces only.
2.4.11 Vectors	
Understand and use vectors in three dimensions.	To include the use of the unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.
Use vectors to solve problems in context, including forces and kinematics.	Questions will not involve the scalar product.

ASSESSMENT OBJECTIVES **AO1: Use and apply standard techniques**	**Weighting**	
	AS	**A Level**
Learners should be able to: • select and correctly carry out routine procedures; and • accurately recall facts, terminology and definitions	**45% – 55%**	**45% – 55%**

Example of a question that is assessing AO1

SAMs Unit 1, Pure Mathematics

1. The circle *C* has centre *A* and equation

$$x^2 + y^2 - 2x + 6y - 15 = 0.$$

(*a*) Find the coordinates of *A* and the radius of *C*. [3]

(*b*) The point *P* has coordinates (4, – 7) and lies on *C*. Find the equation of the tangent to *C* at *P*. [4]

ASSESSMENT OBJECTIVES **AO2: Reason, interpret and communicate mathematically**	**Weighting**	
	AS	**A Level**
Learners should be able to: • construct rigorous mathematical arguments (including proofs); • make deductions and inference; • assess the validity of mathematical arguments; • explain their reasoning; and • use mathematical language and notation correctly.	**20% – 30%**	**20% – 30%**

Example of a question that is assessing AO2

SAMs Unit 1, Pure Mathematics

6. In each of the two statements below, c and d are real numbers. One of the statements is true while the other is false.

   A Given that $(2c + 1)^2 = (2d + 1)^2$, then $c = d$.
   B Given that $(2c + 1)^3 = (2d + 1)^3$, then $c = d$.

   (*a*) Identify the statement which is false. Find a counter example to show that this statement is in fact false.

   (*b*) Identify the statement which is true. Give a proof to show that this statement is in fact true. [5]

ASSESSMENT OBJECTIVES **AO3: Solve problems within mathematics and in other contexts**	**Weighting**	
	AS	**A Level**
Learners should be able to: • translate problems in mathematical and non-mathematical contexts into mathematical processes; • interpret solutions to problems in their original context, and, where appropriate, evaluate their accuracy and limitations; • translate situations in context into mathematical models; • use mathematical models; and • evaluate the outcomes of modelling in context, recognise the limitations of models and, where appropriate, explain how to refine them.	**20% – 30%**	**20% – 30%**

Mathematical Problem Solving

Attributes of a problem solving task:

- Little or no scaffolding.
- Multiple representations, eg. sketch/diagram as well as calculations.
- Information is not given in mathematical form/language; or interpretation of results or evaluation of methods in a real world context.
- Variety of techniques that could be used.
- Solution requires understanding of the processes involved.
- Two or more mathematical processes required or different parts of mathematics to be brought together to reach a solution.

(*A Level Mathematics Working Group Report,* December 2015, Ofqual/15/5789)

Examples of questions that are assessing AO3

SAMs Unit 3, Pure Mathematics

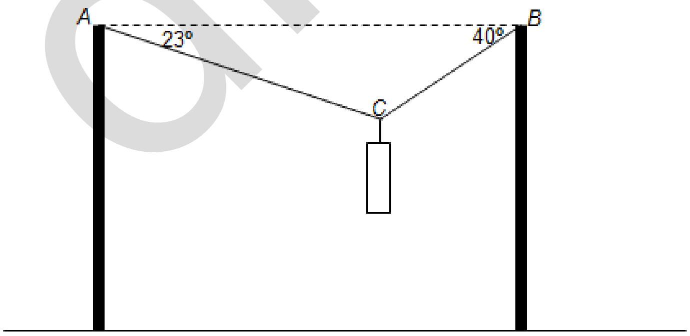
2. Air is pumped into a spherical balloon at the rate of 250 cm^3 per second. When the radius of the balloon is 15 cm, calculate the rate at which the radius is increasing, giving your answer to three decimal places [3]

8. (b) Use an appropriate substitution to show that

$$\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}}\,\mathrm{d}x = \frac{\pi}{12} - \frac{\sqrt{3}}{8}.$$ [8]

SAMs Unit 4, Applied Mathematics, Section B

7. The diagram below shows an object of weight 160 N at a point *C*, supported by two cables *AC* and *BC* inclined at angles of 23° and 40° to the horizontal respectively.



(a) Find the tension in *AC* and the tension in *BC*. [6]

(b) State two modelling assumptions you have made in your solution. [2]

AS Unit 1: Pure Mathematics A

- Trigonometry — (i) Prove trigonometric identities.
- Vectors — (i) Vectors in 2-D.

A2 Unit 3: Pure Mathematics B

- Algebra and Functions — (i) Partial fractions with three linear terms in the denominator.
- Trigonometry — (i) Small angle approximations.
  (ii) Prove trigonometric identities.
- Differentiation — (i) Differentiate $\sin x$ and $\cos x$ from first principles.
  (ii) Differentiate a^{kx}
  (iii) Connected rates of change.
- Numerical Methods — (i) Iterative method $x = g(x)$, cobweb and staircase diagrams.
  (ii) Newton-Raphson method.

Example

1. a) Show that $5\cos^2\theta - 3\sin^2\theta \equiv 8\cos^2\theta - 3$.

   b) Hence solve the equation $5\cos^2\theta - 3\sin^2\theta = -1$ for values of θ between 0° and 180°.

Solution

a) $5\cos^2\theta - 3\sin^2\theta \equiv 5\cos^2\theta - 3(1 - \cos^2\theta)$

$= 5\cos^2\theta - 3 + 3\cos^2\theta$

$= 8\cos^2\theta - 3$.

b) $8\cos^2\theta - 3 = -1$

$8\cos^2\theta = 2$

$\cos^2\theta = \frac{1}{4} \longrightarrow \cos\theta = \pm\frac{1}{2}$

$\theta = 60^\circ, 120^\circ$.

Example

1. A, B, C are three points given by position vectors $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = 14\mathbf{i} - 2\mathbf{j}$, $\mathbf{c} = -8\mathbf{i} + 3\mathbf{j}$ with respect to the origin O.

   a) Find **AB** in terms of $\mathbf{i}$ and $\mathbf{j}$.

   b) Find AB, the magnitude of **AB**.

   c) Find the position vector of *M*, the midpoint of *BC*.

   d) Find the position vector of the point *P* which divides *AC* in the ratio 3 : 7.

Solution

a) $\mathbf{AB} = \mathbf{b} - \mathbf{a} = 14\mathbf{i} - 2\mathbf{j} - (2\mathbf{i} + 3\mathbf{j}) = 12\mathbf{i} - 5\mathbf{j}$.

b) $|\mathbf{AB}| = \sqrt{12^2 + (-5)^2} = 13$

c) *M* is the midpoint of *BC*, therefore

$\mathbf{BM} = \mathbf{MC}$

$\mathbf{m} - \mathbf{b} = \mathbf{c} - \mathbf{m}$

$2\mathbf{m} = \mathbf{c} + \mathbf{b}$ so that $\mathbf{m} = \frac{1}{2}(\mathbf{c} + \mathbf{b})$

$\mathbf{m} = \frac{1}{2}(-8\mathbf{i} + 3\mathbf{j} + 14\mathbf{i} - 2\mathbf{j}) = 3\mathbf{i} + \frac{1}{2}\mathbf{j}$

d) *P* divides *AC* in the ratio 3 : 7 therefore *AP* : *PC* = 3 : 7

$7\mathbf{AP} = 3\mathbf{PC}$

$7(\mathbf{p} - \mathbf{a}) = 3(\mathbf{c} - \mathbf{p})$

$10\mathbf{p} = 3\mathbf{c} + 7\mathbf{a} = 3(-8\mathbf{i} + 3\mathbf{j}) + 7(2\mathbf{i} + 3\mathbf{j})$

$\mathbf{p} = -\mathbf{i} + 3\mathbf{j}$

Example

1. Express $\dfrac{x+1}{(x-1)(x-2)(x-3)}$ in partial fractions.

Solution

$$\frac{x+1}{(x-1)(x-2)(x-3)} \equiv \frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3}$$

$x + 1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$

$x = 1$: $2 = A(-1)(-2)$ $\quad A = 1$

$x = 2$: $3 = B(1)(-1)$ $\quad B = -3$

$x = 3$ $\quad 4 = C(2)(1)$ $\quad C = 2$

$$\frac{x+1}{(x-1)(x-2)(x-3)} = \frac{1}{x-1} - \frac{3}{x-2} + \frac{2}{x-3}$$

Example

1. Use the approximations $\sin x \approx x$, $\tan x \approx x$, $\cos x \approx 1-\frac{1}{2}x^2$ to find the smallest positive root of the equation $\sin x + \cos x + \tan x = 1.5$.

   Give your answer correct to 3 decimal places.

Solution

$$x + 1 - \frac{1}{2}x^2 + x = 1.5$$

$$\frac{1}{2}x^2 - 2x + 0.5 = 0$$

$$x^2 - 4x + 1 = 0$$

$$x = \frac{4 \pm \sqrt{12}}{2} = 3.732 \text{ or } 0.268$$

therefore $x = 0.268$ to 3 decimal places.

Example

1. Show that $\tan A + \cot A \equiv 2\,\text{cosec}\,2A$

Solution

$$\tan A + \cot A = \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}$$

$$= \frac{\sin^2 A + \cos^2 A}{\cos A \sin A}$$

$$= \frac{1}{\sin A \cos A} = \frac{2}{2\sin A \cos A}$$

$$= \frac{2}{\sin 2A} = 2\,\text{cosec}\,2A$$

Examples

1. Differentiate $\sin x$ from first principles.

Solution

Let $f(x) = \sin x$

$f(x + h) = \sin(x + h) = \sin x \cos h + \sin h \cos x$

when h is very small $\sin h \approx h$ and $\cos h \approx 1 - \frac{1}{2}h^2$

so that $f(x + h) = \sin x(1 - \frac{1}{2}h^2) + h \cos x = \sin x - \frac{1}{2}h^2 \sin x + h \cos x$

$f(x + h) - f(x) = -\frac{1}{2}h^2 \sin x + h \cos x$

$$\frac{f(x+h) - f(x)}{h} = -\frac{1}{2}h \sin x + \cos x$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x) = \cos x$$

2. Differentiate 5^x with respect to x.

Solution

Let $f(x) = 5^x$

Taking logs: $\ln f(x) = \ln 5^x$

$\ln f(x) = x \ln 5$

differentiating $\dfrac{f'(x)}{f(x)} = \ln 5$

$f'(x) = f(x) \ln 5 = 5^x \ln 5$

3. The area, A cm^2, of a circular blot of ink is increasing at a rate of 3 cm^2/sec. Find the rate at which the radius is increasing when the radius is 2 cm, giving your answer correct to 2 decimal places.

Solution

Since $A = \pi r^2$

$$\frac{\mathrm{d}A}{\mathrm{d}r} = 2\pi r$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}A} \times \frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2\pi r} \times 3 = \frac{3}{2\pi r}$$

When r = 2

$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{3}{4\pi}$ = 0.24 cm/sec to 2 decimal places.

Examples

1. The equation $e^x + x - 3 = 0$ has a positive root α.

   a) Show that α lies between 0.5 and 1.

   b) Use the iterative formula $x_{n+1} = \ln(3 - x_n)$ and $x_0 = 1$ to find α correct to 3 decimal places.

   c) Use the Newton-Raphson method to find α correct to 5 decimal places.

Solution

a) Let $f(x) = e^x + x - 3$

$f(0.5) = -0.851278\ldots < 0$

$f(1) = 0.718281\ldots > 0$

change of sign, therefore the root α is between 0.5 and 1.

b) Using $x_{n+1} = \ln(3 - x_n)$ and $x_0 = 1$

$x_1 = 0.6931471806$ ..

$x_2 = 0.8358841798$ ..

$x_3 = 0.7720118809$ ..

$x_4 = 0.8010989895$ ..

$x_5 = 0.7879576949$ ..

$x_6 = 0.7939162088$ ..

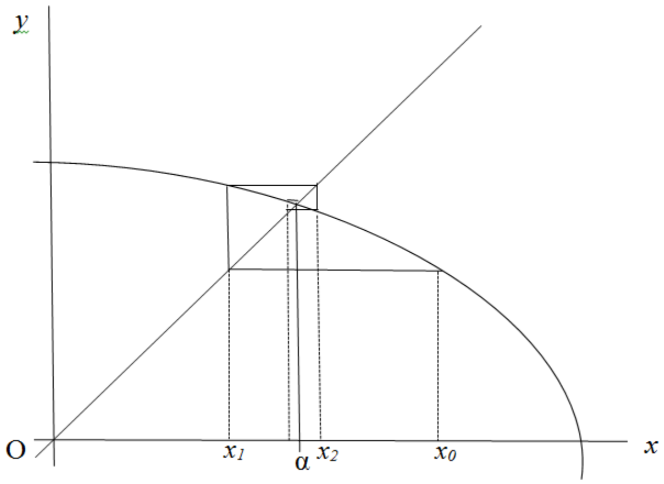
$x_7 = 0.7912189034$ ..

$x_8 = 0.7924408234$ ..

$x_9 = 0.7918874602$..

therefore $\alpha = 0.792$ to 3 decimal places.

This cobweb diagram illustrates how the values converge towards the root α.



c) Using Newton-Raphson:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{e^{x_n} + x_n - 3}{e^{x_n} + 1}$$

$x_0 = 1$

x_1 = 0.8068242641 ..

x_2 = 0.7921349597 ..

x_3 = 0.7920599704 ..

x_4 = 0.7920599684 ..

therefore α = 0.79206 to 5 decimal places.